Limit Theorems for Limit Order Books

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Abstract

For several decades now, a substantial part of securities trading has been carried out using electronic limit order books (LOBs). An LOB exists as information on a server at an exchange and is usually accessible to all market participants. It displays the current supply and demand of a security at discrete prices and trading occurs at discrete times for a discrete number of shares. The generic model for LOBs is random, state-dependent and high dimensional since orders may be placed at many different prices. These features lead to an inherent mathematical complexity in order book modeling. There are still quite few qualitative models of limit order books which focus on rigorous mathematical analysis. The approach in this thesis consists of three steps: i) Defining a random discrete model that describes the order book dynamics. ii) Scaling the original discrete model by letting the arrival rate tend to infinity, the volumes go to zero and number of possible prices become infinitely (continuously) many. iii) Proving limit theorems for the scaled models and identifying the limiting model.

In the first part of the thesis, we define a random state-dependent discrete model of a two-sided limit order book in terms of its key quantities *best bid [ask] price* and the *standing buy [sell] volume density*. For a simple scaling that introduces a slow time scaling, that is equivalent to the classical law of large numbers, for the bid/ask prices and a faster time scale for the limit volume placements/cancelations, that keeps the expected volume rate over the considered price interval invariant, we prove a limit theorem. The limit theorem states that, given regularity conditions on the random order flow, the key quantities converge in the sense of a strong law of large numbers to a tractable continuous limiting model. The limiting model is such that the best bid and ask price dynamics can be described in terms of two coupled ODE:s, while the dynamics of the relative buy and sell volume density functions are given as the unique solutions of two linear first-order hyperbolic PDE:s with variable coefficients, specified by the expectation of the order flow parameters. The solutions may be given in closed form and we provide a worked out example for the classical assumption of Poisson arrivals. An alternative scaling, which may be advantageously used to show a weak law of large numbers, is also provided.

In the second part, we prove a functional central limit theorem i.e. an invariance principle for an order book model with block shaped volume densities close to the spread. Here, the best bid and ask price processes are jointly considered as a two-dimensional process where the x-variable models the best bid price and the y-variable models the best ask price. The weak limit of the price process is given by a semi-martingale reflecting Brownian motion in the set of admissible prices. The reflections on the boundary, occur naturally when the best bid and ask price coincide and when the bounds of the price interval are reached. Simultaneously, the relative buy and sell volume densities close to the spread converge weakly to the modulus of a 2-parameter Brownian motion. We also demonstrate an example how to easily derive an SPDE for the relative volume densities in a simple case, when a strong stationarity assumption is made on the limit order placements and cancelations for the model suggested in the first part.

In the third and final part of the thesis, we prove an averaging principle and an invariance principle for discrete processes taking values in Banach and Hilbert spaces, respectively.

Key words: limit order book, law of large numbers, functional central limit theorem, invariance principle, scaling limit, averaging principle, queueing theory

Zusammenfassung

Seit mehreren Jahren werden Wertpapiere zumeist über elektronische Limit Orderbücher (LOBer) gehandelt. Ein LOB existiert als Information auf einem Server eines Handelsplatzes, wo es meistens für sämtliche Marktteilnehmer abrufbar ist. Es stellt das Angebot und die Nachfrage für ein bestimmtes Wertpapier zu diskreten Preisen dar. Der Handel findet jeweils zu einer diskreten Stückzahl statt. Das übliche Modell eines LOBs ist stochastisch, zustandsabhängig und hochdimensional, da Order (Handelsaufträge) zu vielen verschiedenen Preisen gegeben werden können. Jene Eigenschaften von LOBern tragen dazu bei, dass Orderbuchmodelle komplexe, mathematische Objekte sind. Zur Zeit gibt es nur wenige qualitative Orderbuchmodelle, die Ergebnis strikt mathematischer Überlegungen sind. Der Ansatz in der vorliegenden Dissertation besteht aus drei Teilen: i) Der Definition eines diskreten stochastischen Modells, das die Orderbuchdynamik beschreibt. ii) Der Skalierung des Ursprungsmodells, bei der die Ankunftsraten im Limes gegen unendlich, die Volumina gegen 0 und die Anzahl der Preise unendlich viele (stetig) werden. iii) Dem Beweisen von Grenzwertsätzen und dem Erfassen der Limesmodelle.

Im ersten Teil der Dissertation wird ein diskretes stochastisches zustandsabhängiges Modell eines zweiseitigen LOBs als bestehend aus den Zustandsgrößen bester Bidpreis (Geldkurs), bester Askpreis (Briefkurs) und vorhandener Kauf- bzw. Verkaufsdichte definiert. Für eine einfache Skalierung mit zwei Zeitskalen wird ein Grenzwertsatz bewiesen. Die Veränderungen der besten Bid- und Askpreise werden im Sinne des Gesetzes der großen Zahlen skaliert und dies entspricht der langsameren Zeitskala. Das Platzieren bzw. Stornieren der Limitorder findet auf der schnelleren Zeitskala statt. Der Grenzwertsatz besagt, dass die fundamentalen Zustandsgrößen, gegeben Regularitätsbedingungen der einkommenden Order, fast sicher zu einem stetigen Limesmodell konvergieren. Im Limesmodell sind der beste Bidpreis und der beste Askpreis die eindeutigen Lösungen von zwei gekoppelten gewöhnlichen DGLen. Die Kauf- und Verkaufsdichten sind jeweils als eindeutige Lösungen von linearen hyperbolischen PDGLen, die anhand der Erwartungswerte der einkommenden Orderparameter festgelegt sind, gegeben. Die Lösungen sind in geschlossener Form erhältlich. Für poissonverteilte Orderankünfte wird ein Beispiel berechnet. Eine alternative Skalierung kann vorteilhafterweise verwendet werden um ein schwaches Gesetz der großen Zahlen zu beweisen.

Im zweiten Teil wird ein funktionaler zentraler Grenzwertsatz d.h. ein Invarianzprinzip auf einem geschlossenen Preisintervall für ein vereinfachtes Modell eines Limitorderbuches bewiesen. Hierbei werden der Bid- und Askpreis als zweidimensionaler Prozess gemeinsam betrachtet, wobei die x-Variable den besten Bid- und die y-Variable den besten Askpreis bezeichnen. Unter einer natürlichen Skalierung konvergiert der Prozess in Verteilung zu einer Semimartingal reflektierten Brownschen Bewegung in der zugelassenen Preismenge. Die Reflektionen auf den Rändern finden statt, wenn der beste Bidpreis und der beste Askpreis übereinstimmen und wenn die untere Preisgrenze erreicht wird. Gleichzeitig konvergieren die Kauf- und Verkaufsdichten im schwachen Sinn zum Betrag einer zweiparametrischen Brownschen Bewegung. Es wird weiterhin anhand eines Beispiels gezeigt, wie man für das Modell im ersten Teil eine stochastiche PDGL, unter einer starken Stationaritätsannahme für die Orderplatzierungen und -stornierungen, herleiten kann.

Im dritten Teil wird ein Mittelungsprinzip bzw. ein Invarianzprinzip für diskrete Banach- bzw. Hilbertraumwertige stochastische Prozesse bewiesen.

Schlagwörter: Limit Orderbuch, Gesetz der großen Zahlen, Invarianzprinzip, Skalierungslimit, Mittelungsprinzip, Warteschlangentheorie

Contents

1 Introduction

This thesis is concerned with the stochastic modeling and approximation of the market microstructure that arises in electronic limit order books (LOBs). From the perspective of mathematical finance it is both relevant and interesting to study LOBs, since a large part of transactions in the financial markets are carried out using these market structures which, at the same time, exhibit an inherent mathematical complexity. In the following, we describe how an electronic LOB works in practice, present a survey of related work and provide a summary of the results of the thesis.

1.1 The Microstructure of Electronic Limit Order Books

An electronic LOB exists as information on a server at an exchange and is usually accessible to all market participants. It displays the current supply and demand of a security at discrete prices and trading occurs at discrete times for a discrete number of shares. The discreteness property follows since the LOB is implemented on a computer which, *by construction*, runs on a finite arithmetic. Although electronic exchanges have been around for little over 40 years, the basic market mechanism behind the limit order book, the *continuous double auction*, has a long history which goes back to traditional trading pits and open-outcry markets[1]. We proceed to describe how the underlying market mechanisms of these exchanges work in practice i.e. the microstructure of electronic LOBs.

The state of an LOB is the result of an order flow and the matching of those orders according to trading rules. Orders are effectively trading instructions and there are typically two categories in an LOB: *limit orders* and *market orders*. A buy [sell] limit order specifies the volume i.e. the number of shares, contracts or units of the security or

[1] According to Coppejans and Domowitz [19, p.1]: "The continuous double auction, in particular, has been the dominant financial market structure in the U.S. for over 140 years, under the rubric of open-outcry floor trading". To the best of our knowledge, the first completely electronic stock exchange was the NASDAQ which opened in 1971, see Föllmer and Schied [32]. Today, most exchanges are not only electronic but highly computerized. Algorithmic trading (automated and mostly very frequent submission of trading instructions by market participants using computerized trading strategies) has shown to play an increasingly important part in markets. Recent reports indicate that nearly 40% of all share orders in Europe and 37% in the U.S. are sent by algorithmic trading computers, see Stothard [73].

asset to be bought [sold] at an upper [lower] price limit. If there is some sell [buy] volume available, below [above] the specified price limit in the order book, a trade occurs for the matching volume. The remaining part of the placed limit order is collected in the electronic order book awaiting execution. A market buy [sell] order specifies only the volume to be bought [sold] and is matched with standing sell [buy] limit order volume at the lowest [highest] available price in the LOB. The highest [lowest] price at which there is standing buy [sell] limit order volume is a key quantity and is called the best bid [ask] price. Usually, standing limit orders may be canceled, by the market participant who placed the order, at any moment. See Figures 1.1-1.3 for illustrations of these order events[2] for the buy side (the sell side events can be illustrated analogously).

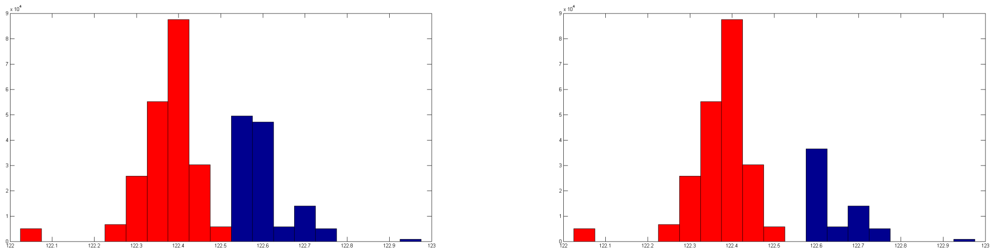

Figure 1.1: The state of the order book before (left) and after (right) a market buy order arrival.

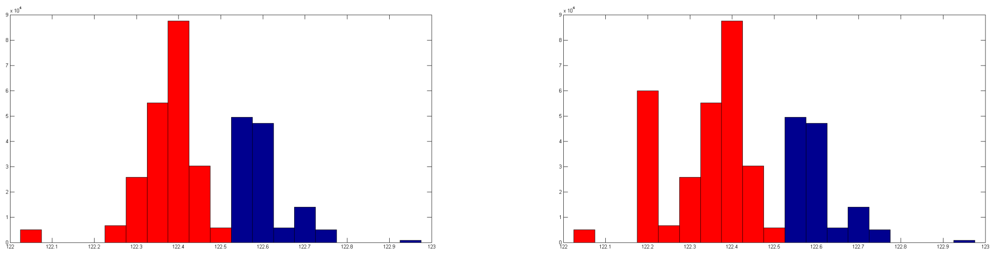

Figure 1.2: The state of the order book before (left) and after (right) a buy limit order arrival with limit price 122.20 i.e. 5 price levels below the best bid price.

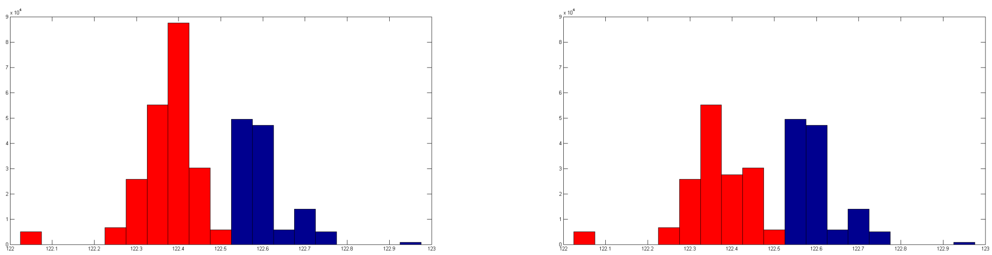

Figure 1.3: The state of the order book before (left) and after (right) the cancelation of a buy limit order at 122.40 i.e. 2 price levels below the best bid price.

[2]We also want to point out that exchanges typically offer a large variety of orders that are combinations of the mechanisms just described as well as *hidden orders*, see Harris [40, p.82-86].

Standing limit order volumes at the best bid and ask prices may be seen as *first in line* for being matched with incoming (effective) market orders. Thus, an order book is a two sided queueing system, where expensive buy limit orders and cheap sell limit orders are given price priority. Limit orders placed at the same price level are matched on a First-In-First-Out basis. The difference "best ask price - best bid price" is called the spread. There is a minimum price increment in an order book, called the price tick. Thus, if the spread is larger than one price tick it is possible to *step first in line* by submitting a buy [sell] order above [below] the best bid [ask] price. Similarly, there is a minimum inter arrival time of orders called the time tick. The tick sizes are a measure of the *discreteness* of an order book and have a profound effect on its dynamics. This could be observed after the decimalization of the North American equity markets. In the case of U.S. equity markets in 2001, the price tick was reduced from 1/16 of a dollar to 1 cent. The value of time-priority was greatly reduced as it became easier to overtake orders on a price basis, orders became smaller and more frequent, see Harris [40, p.115]. Another example of relative tick size effects (price tick/price), often quoted by practitioners, are the order book dynamics of Intel Corporation (INTC) and Apple Inc. (AAPL) traded at the NASDAQ, see the plots of intra-day data[3] in Figures 1.4-1.7. The tick size is $0.01 for both stocks but relatively larger in the case of INTC than in the case of AAPL and this is sometimes given as an explanation for the clustering of volumes close to the spread.

1.2 Modeling and Related Work

The generic model for order books is random, state-dependent and high dimensional since orders may be placed at many different prices. These features lead to an inherent mathematical complexity. Consequently, the models suggested in the vast econophysics and econometrics literature have had more of a quantitatively statistical approach. There is also a significant economic and econometric literature on LOBs including Biais at al. [11], Easley and O'Hara [25], Foucault et al. [33], Glosten and Milgrom [36], Parlour [62], Rosu [67] and many others that puts a lot of emphasis on the realistic modeling of the working of the LOB, and on its interaction with traders' order submission strategies. There are also many works on optimal liquidation in limit order books, see e.g. the references given in the survey by Föllmer and Schied [32].

To the best of our knowledge, there are still very few qualitative models of limit order books whose focus is on a rigorous mathematical analysis. We now give a short summary of some interesting papers within this field of research. Among the first to appear was the one by Kruk [53], who studied a queueing theoretic model of a transparent double

[3]The data was generously provided by LOBSTER [41], the order book data reconstruction service recently launched at the Humboldt-Universität zu Berlin.

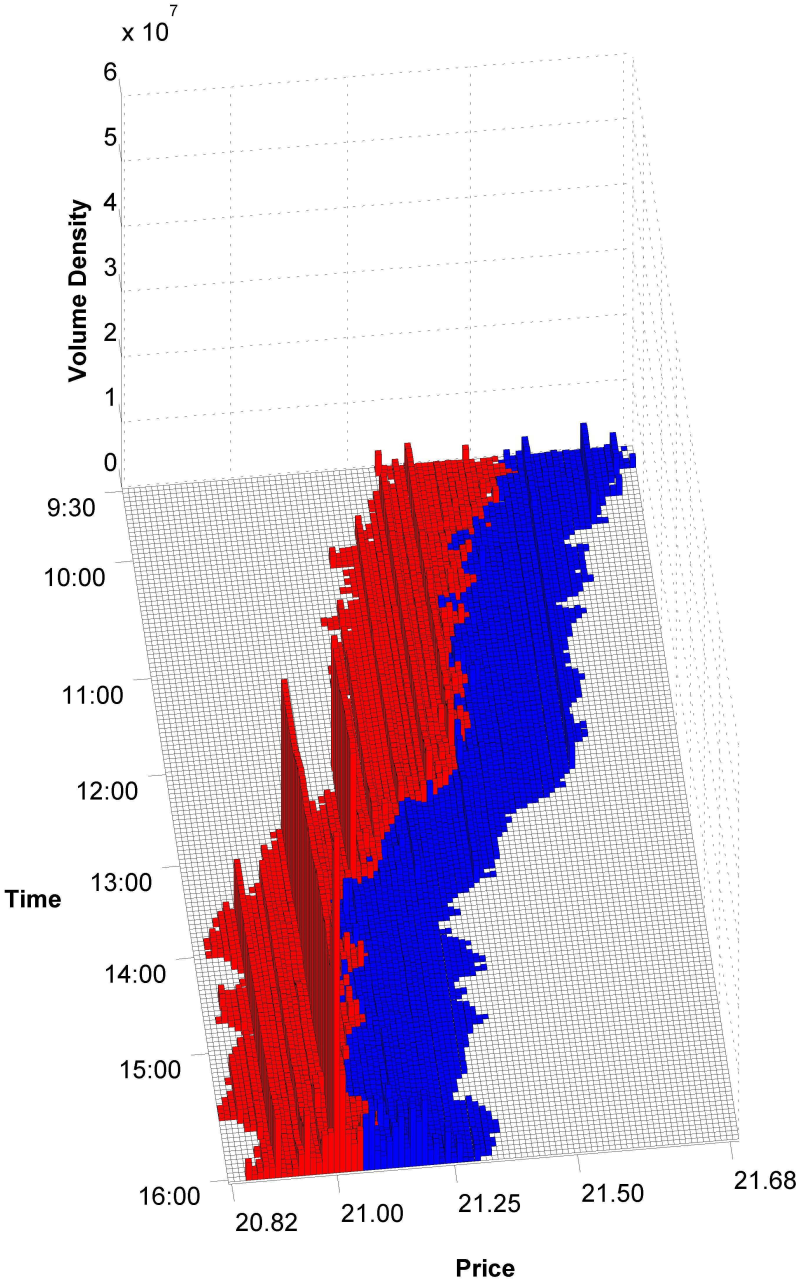

Figure 1.4: The order book of INTC from the 4th of April 2013, 20 price levels from the spread, sampled every 3000 events.

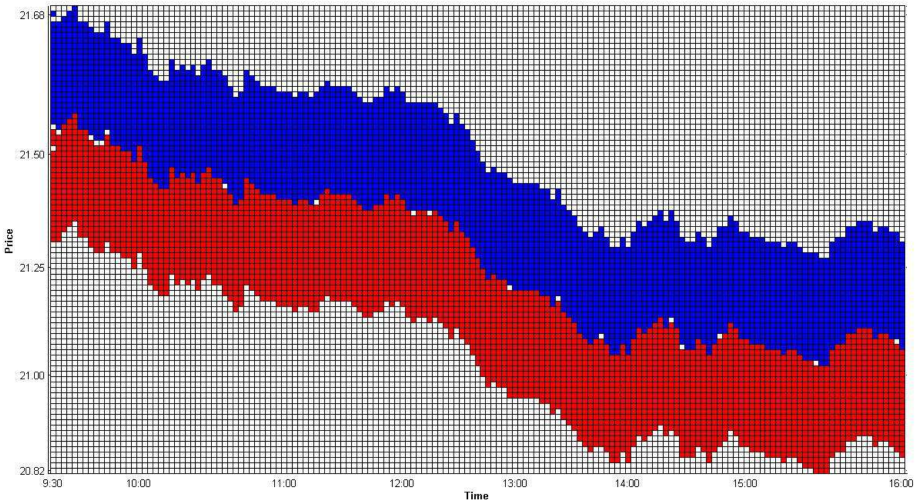

Figure 1.5: The plotted price evolution of the INTC order book in Figure 1.4.

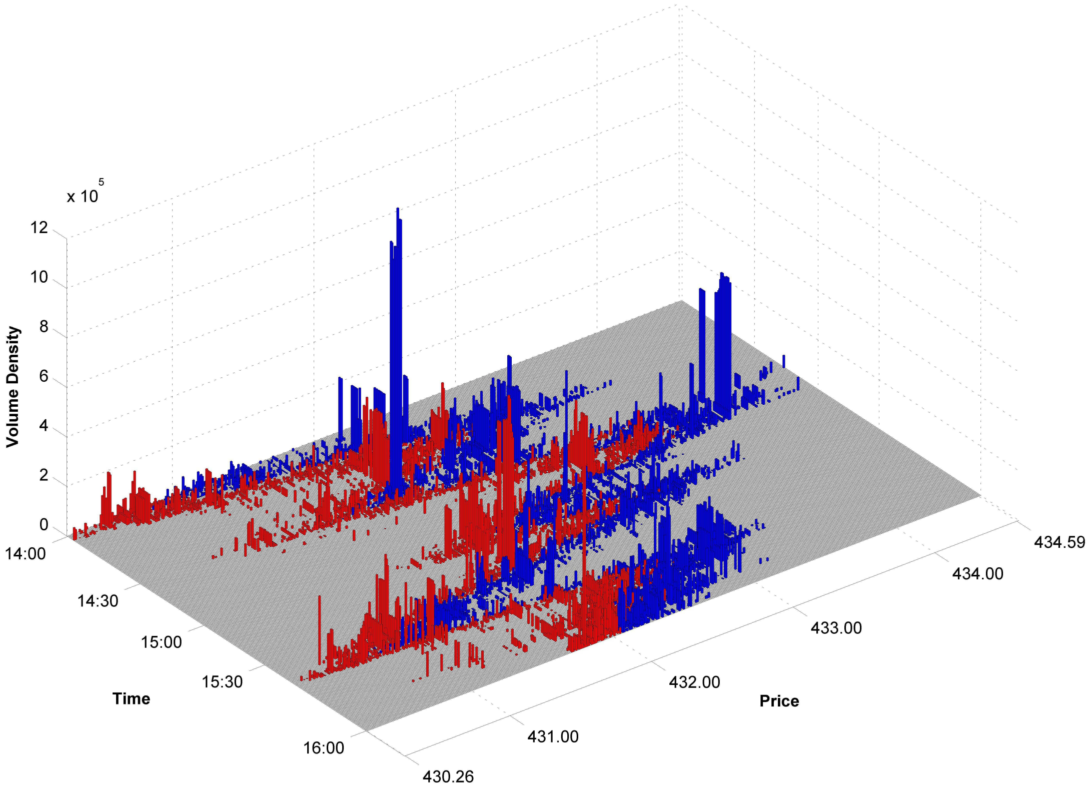

Figure 1.6: The order book of AAPL from the 4th of April 2013 between 14:00 and 16:00, 20 price levels from the spread, sampled every 300 events.

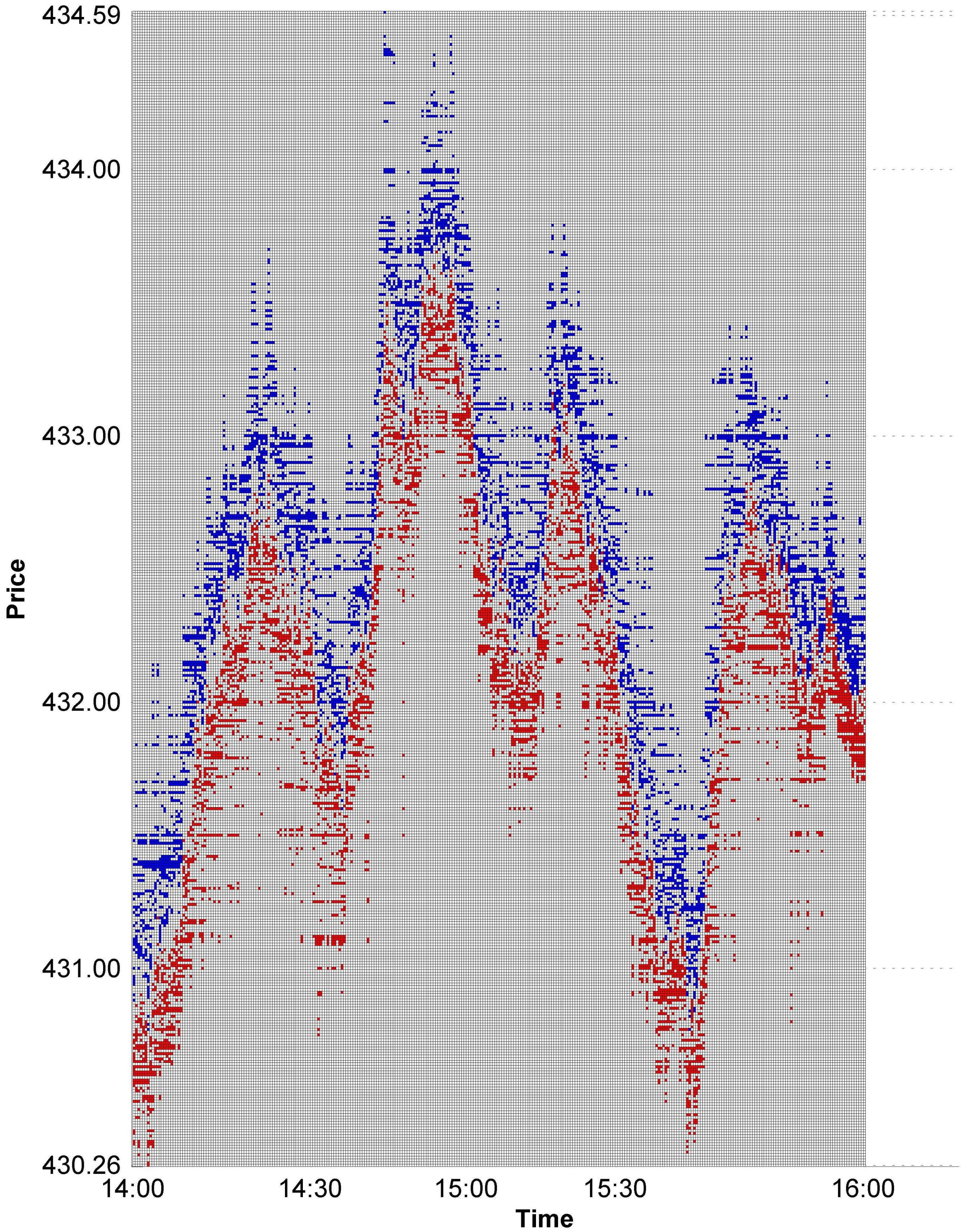

Figure 1.7: The plotted price evolution of the AAPL order book in Figure 1.6.

auction in continuous time. The microstructure may be interpreted as that of a simple limit order book, if one considers a buyer as a buy limit order and a seller as a sell limit order. At the auction, there are $N \in \mathbb{N}$ possible prices of the security and thus $2N$ different classes of customers (N classes of buyers and N classes of sellers) arriving at a queueing network that is First-In-First-Out within a class and where priority for trading at the auction is given to the buyer [seller] currently willing to pay the highest [lowest] of the N possible prices. The customer arrivals are modeled as renewal processes, satisfying certain integrability conditions. The order volume of a customer is random and there are no cancelations. Taking scaling limits under heavy-traffic conditions, Kruk was able to show a diffusion limit (assuming $N = 2$ possible prices) and a fluid limit (assuming $N \geq 3$ possible prices). The diffusion limit states that for $N = 2$ possible prices, the scaled number of outstanding buy orders at the lower price and the scaled number of outstanding sell orders at the higher price converge weakly to a semimartingale reflected two-dimensional Brownian motion in the first quadrant. The fluid limit states that several key quantities (e.g. the scaled number of outstanding orders and the scaled number of immediately traded shares) converge weakly to affine functions of time, where the coefficients of the functions are given by the input parameters of the model (e.g. the probability for an arriving customer to belong to a certain class, expected inter arrival time and the expected number of shares a customer immediately trades upon arrival). This means that typically, the auction never clears and the various volumes increase linearly in time at a constant rate, which is quite intuitive under heavy-traffic conditions without any cancelation and where the flow of customers is assumed to be independent of the current state of the auction/order book.

Bovier and Černy [13] studied a two-species interacting particle model on a subset of $\mathbb{Z}$ with open boundaries. The two species are injected with time dependent rate on the left, resp. right boundary. Particles of different species annihilate when they try to occupy the same site, similar to the matching of buy and sell orders. Thus, the model has been proposed as a simple model for the dynamics of an LOB. The authors considered the hydrodynamic scaling limit for the empirical process and proved a large deviation principle that implied convergence to the solution of a deterministic non-linear parabolic equation.

A very interesting qualitative order book model was suggested by Osterrieder [60]. Here, one assumes an exogenously given price process and the order book is modeled as several measure-valued processes. The mathematical concepts in use come from the theory of random measures and point processes. The framework is quite flexible and the central assumptions are the following: i) A small investor model is considered, i.e. the price impact of orders, since they are so small in volume, is negligible. ii) There exists an exogenously given reference price (in effect a transaction price) for the stock and it is modeled as a semimartingale stochastic process in continuous time (sometimes a general Itô process or a Geometric Brownian motion). iii) The limit orders are submitted relative to the reference price such that all buy (sell) orders are placed below (above) the reference price. iv) A random measure describes the number of orders which arrive in a certain subset of a three-dimensional space which is spanned by time, relative limit order price and limit order size. v) The orders in the limit order book are executed once they are reached by the reference price process. vi) Cancelation is possible. Under these assumptions, it turns out that the order book can be described as the difference of two doubly stochastic Poisson processes (the buy and sell side) at every point in time. There is also an alternative characterization which can be given as the difference of two infinite sums of Bernoulli distributed random variables. One can calculate the distribution of the execution probability of limit orders and the time horizons of limit order traders. The intensity measure may be considered as an exotic option (a reverse Asian fixed strike lookback option) and one can find the distributions of the traded volume. The setting is essentially that of a small investor model and Osterrieder gives some suggestions on how to expand the model to make it a large trader model. The ideas turn to the usual setting of stochastic finance (no arbitrage assumption, admissible and self-financing strategies, etc.) and defines the trading strategy of a large trader.

Since information on the best bid and ask price and volume at different price levels of an order book are available to all market participants, it is indeed realistic to assume that the dynamics depend on the current state of the order book. The feature of conditional state-dependence was considered by Cont et al. [17], who proposed a continuous time stochastic model with a finite number of possible prices for the security and where the events (buy/sell market order, buy/sell limit order placement and cancelation) are modeled using independent Poisson processes. The arrival rates of the limit orders depend on the distance to the best/bid and ask in a power law fashion. The authors were able to show that the state of the order book, defined as a vector containing all volumes in the order book at different prices, is an ergodic Markov process that has a proper stationary distribution. Using this fact, several interesting quantities could be calculated using Laplace transforms, conditional on the current state of the order book without taking any scaling limits, e.g. the probability of a mid price move, a move in the bid price before a move in the ask price and volume execution before a price move happens. These quantities were computed, compared to empirically observed values and found to fit well with the short-time behavior of the order book.

In the paper by Cont and de Larrard [16] a scaling limit in the diffusion sense was considered for a Markovian Limit Order Market in which the state is represented by the best bid and ask price and the queue length i.e. the number of outstanding orders at the best bid and ask price, respectively. With this reduction of the state space, under symmetry conditions on the spread and stationarity conditions on the queues at the best bid and ask, it was shown that the price (which may be interpreted as the transaction price) converges to a Brownian motion without drift and with volatility given by the model parameters in the diffusion limit. Very recently, Cont and de Larrard [15] studied the reduced state space under weaker conditions and were able to prove a refined diffusion limit by showing that under heavy traffic conditions the bid and ask queue lengths are given by a two-dimensional Brownian motion in the first quadrant with reflection to the interior at the boundaries, similar to the diffusion limit result for $N = 2$ prices in Kruk [53, Theorem 4.4 on p.727]. The price becomes a càdlàg process which increases by one tick if the ask queue hits zero and decreases by one tick if the bid queue hits zero.

In the framework analyzed by Abergel and Jedidi [1], the volumes of the order book at different distances to the best bid and ask were modeled as a finite dimensional continuous time Markov chain and the order flow as independent Poisson processes. Under the assumption that the width of the spread is constant in time, using Foster-Lyapunov stability criteria for the Markov chain, the authors proved ergodicity of the order book and a diffusion limit for the mid price. In the diffusion limit, the mid price is a Brownian motion with constant volatility given by the averaged price impact of the model events on the order book.

1.3 Summary of Obtained Results

The approach of the thesis can be summarized as follows:

i) Defining a random discrete model that describes (as much as possible of) the microstructure of the limit order book. This corresponds to the microscopic setting of the LOB dynamics.

ii) Scaling the original discrete model by considering a sequence of scaled discrete models and letting the arrival rate tend to infinity, the volumes go to zero and the number of possible prices become infinitely (continuously) many.

iii) Proving limit theorems for the scaled models and identifying the limiting model i.e. the mesoscopic and macroscopic limits.

The three steps are of course very much inter-dependent and different scalings typically yield different convergence modes and limit models, see Table 1.1 for an overview of the scalings and limits of the thesis.

1.3.1 Chapter 2: Laws of Large Numbers for LOBs

In the first part of the chapter, we prove a strong law of large numbers for LOBs. We propose a discrete continuous-time model of a two-sided state-dependent order book with random order flow and cancelation with countably many price levels. The buy and the sell side volumes are coupled through the best bid and ask price dynamics. We model the buy and sell side volumes as density functions in relative price coordinates, i.e. relative to their distance from the best bid/ask prices. The integral of the relative buy [sell] volume density function between two price levels is equal to the buy [sell] volume at the lower of the prices. Volumes at positive distances are defined as standing limit volume, observable in the order book ("visible book"), and volume at negative price distances corresponds to volume that would be placed in the spread if the next order would be a spread limit order ("shadow book"). The state of the model at any point in time is thus a quadruple comprising the best bid price, the best ask price, the relative buy volume density function and the relative sell volume density function. The state-dynamics is then conveniently defined in terms of a recursive stochastic process taking values in a function space.

To establish our scaling limit we use a mathematical framework similar to the recurrently defined semi-Markov processes on $\mathbb{R}^n$, as in Anisimov [5, p.95-116], [6, 7] or Gikhman and Skorokhod [35, p.184-208]. However, in our case the stochastic processes take values in $\mathbb{R}^2$ and $L^2(\mathbb{R}, \mathbb{R})$. Furthermore, our scaling is highly nonlinear. When the analysis of the market is limited to prices as in e.g. Bayraktar et al. [9], Garman [34] or Horst and Rothe [46] or to the joint dynamics of prices and *aggregate volumes* (e.g. at the top of the book) as in Cont and de Larrard [15, 16] the limiting dynamics can naturally be described by ordinary differential equations or real-valued diffusion processes, depending on the choice of scaling[4]. The analysis of the entire LOB including the *distribution* of standing volume across many price levels is much more complex. Osterrieder [60] also modeled both sides of LOBs using measure-valued diffusions. Our approach is based on an averaging principle for Banach space-valued processes, see Chapter 4. The scaling limit requires two time scales: a fast time scale for cancelations and limit order placements outside the spread (passive events that do not lead to price changes), and a comparably slow time scale for market order arrivals and limit order placements in the spread (active events that lead to price changes). The choice of time scales captures the fact that in LOB markets significant proportions of limit orders are never executed, see e.g. Hautsch and Huang [42]. Mathematically, the different time scales imply that the noise of aggregate placements/cancelations is of the same order as the noise of the price changes. Given the classical law of large numbers scaling for the bid/ask prices, we show optimality for the fast time scaling using a martingale difference array extension of the convergence rate in the Brunk-Prokhorov Law of Large Numbers Theorem by Shue et al. [71, Theorem 2.2 on p.3190]. It then holds that the function-valued volume density noise a.s. converges to zero, by the Strong Law of Large Numbers for martingale difference sequences in Banach spaces by Hoffman-Jorgensen and Pisier [43, Theorem 2.2 on p. 591].

The main result of the first part of the chapter states that when limit buy [sell] orders are placed at random distances from the best bid [ask] price and the price tick tends

[4] We refer to Mandelbaum et al. [58] for general approximations results for queueing systems.

to zero, order arrival rates tend to infinity, and the volume placed/canceled tends to zero, then the sequence of scaled order book models converges almost surely uniformly over compact time intervals to a deterministic limit. The limiting model is such that the best bid and ask price dynamics can be described in terms of two coupled ODE:s, while the dynamics of the relative buy and sell volume density functions are given as the unique solutions of two linear first-order hyperbolic PDE:s with variable coefficients. The solutions can be given in closed form. As a corollary to the main result, one gets a simple expression for the observable standing volume density in the limit, in terms of the key quantities of the main result. Thus, our work provides a simple first order approximation i.e. a macroscopic limit model for the entire volume dynamics of an LOB.

The second part of Chapter 2 is based on the paper by Horst and Paulsen [45], where a Weak Law of Large Numbers (WLLN) for LOB:s was shown. The model has many dynamic features in common with the model used in the first part of Chapter 2. The scaling may advantageously be used to show convergence in probability, uniformly over compact time intervals. Many of the proofs and results, that are needed to show almost sure convergence may be used to show the convergence in probability. However, some parts of the proof simplify and can be considerably shortened by utilizing a uniform result by Pisier (Lemma 5.2.7) for martingale difference sequences in Banach spaces.

We conclude the chapter with discussing possible applications of the limiting model and giving an outlook for extensions.

1.3.2 Chapter 3: Functional Central Limit Theorems for LOBs

We consider a simple discrete LOB model for a finite time interval and define the dynamics of the discrete best bid and ask prices. When one considers these prices as the x- and y-variable of a two-dimensional price process, one has a random walk in the wedge defined by the natural conditions that the best bid price is smaller than the best ask price and that both prices are positive. At the boundary of the wedge, the price process is reflected due to the assumptions that: i) There always exist buyers who buy the security at the minimum price of one tick. ii) When the spread is at its minimum of one price tick, there always exist spread traders who buy and sell the volume at the best bid and ask price.

The buy and sell volume densities are defined again using the notion of a *visible book* for standing volumes and a *shadow book* for the spread volumes. The model is simpler than the one considered in Chapter 2 and describes only the volume close to the spread, which is assumed to be block-shaped[5] in the original discrete model. For this purpose, we model the volumes as the modulus of a simple random martingale difference field.

[5] This is a common assumption in many of the models used for optimal trading in LOBs.

Our main result yields a representation for the price processes and infinitely many prices. Under a suitable scaling, the weak limit of the two-dimensional price process is a semi-martingale reflecting Brownian motion (SRBM) in the wedge shaped domain of admissible prices with constant reflection fields at the boundary. The volume densities close to the spread converge to the modulus of a Brownian sheet on $[0,1]\times[0,1]$, respectively. The proofs for the best bid and ask price convergence rely on the general invariance principle in convex polyhedrons of Kang and Williams [51]. We apply the invariance principle of Poghosyan and Rœlly [64] for the convergence of the volume densities. SRBMs in the positive orthant have in the past been found as limits of various queueing systems (see the survey of Williams [77]) and lately describing volumes at the spread e.g. in the works by Kruk [53] and Cont and de Larrard [16]. To the best of our knowledge, SRBMs have not been used to describe the actual prices in an LOB. We believe that our approach could be quite fruitful for reasons of tractability as there exist many interesting results for SRBM in wedges, see e.g. Dieker and Moriarty [23]. On the other hand, we do not model the coupling of prices and volumes explicitly but give mathematical assumptions under which our invariance principle holds. We conclude the chapter with an outlook how this and other extensions e.g. limit theorems under near epoch dependence and subordination assumptions could be achieved.

In the second part of the chapter, we show by example how to derive an SPDE for the model in the first chapter under strong stationarity assumptions on the volume density fluctuations. This is done by proving an invariance principle for the prices, applying the generalized continuous mapping principle in Whitt [76] and the Itô formula. The resulting SPDE is parabolic and of second order.

1.3.3 Chapter 4: Limit Theorems in Banach Spaces

In the final chapter of the thesis, we prove an averaging principle for a discrete stochastic process. The considered process is a generalization of the original discrete processes of Chapters 2 and 3. It is random in state and time, recursively defined and takes values in Banach spaces. The averaging principle states that for a simple linear scaling, the scaled process observed in continuous time converges almost surely to the unique solution of a deterministic ODE on the state space, assumed to be a p-uniformly smooth Banach space where $p \in (1,2]$. These spaces include Hilbert spaces, several types of Sobolev spaces but e.g. not the L^1 or C^1 function spaces. The geometric restriction is needed since we apply the Strong Law of Large Numbers for martingale difference series by Hoffman-Jorgensen and Pisier [43]. To ensure that the ODE has a unique solution, we assume that the state and time operators, used to define the dynamics of the process, are Lipschitz continuous. Our result is a generalization of the semi-Markov setting in $\mathbb{R}^n$ described in Anisimov [5, 6, 7] and Gikhman and Skorokhod [35] for these scalings. The method of proof in Section 4.2 is principally that of Chapter 2.

For a mesoscopic scaling of the state-process in the first section, we state sufficient assumptions for an invariance principle to hold in Hilbert space. The martingale difference case, using the invariance principle for the Robbins-Monro Process by Walk [75], yields a Brownian motion in Hilbert space. We discuss the near-epoch dependence case, using the FCLT for heterogenous arrays by Chen and White [14]. Thus, in these abstract spaces, we consider processes, scalings and assumptions relevant to those of Chapters 2 and 3.

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Chapters 2 and 3:	**SLLN:**	**FCLT:**
inter-arrival time scaling	$\mathcal{O}(\frac{1}{n})$ (active orders) $\mathcal{O}(\frac{1}{n^s})$, $s \in (3, \infty)$ (passive orders)	$\mathcal{O}(\frac{1}{n})$ $\mathcal{O}(\frac{1}{n^2})$
price tick scaling	$\mathcal{O}(\frac{1}{n})$	$\mathcal{O}(\frac{1}{\sqrt{n}})$
volume scaling	$\mathcal{O}(\frac{1}{n^s})$, $s \in (3, \infty)$ (passive orders)	$\mathcal{O}(\frac{1}{n^{3/2}})$
state space, E	$\mathbb{R}^2 \times \big(L^2(\mathbb{R}, \mathbb{R}_+)\big)^2$	$\mathbb{R}^2_+ \times \big(L^2([0,1], \mathbb{R}_+)\big)^2$
convergence	almost sure	in distribution
limit object, prices	solution of coupled ODE:s	SRBM in a wedge
limit object, volumes	volume densities are $C^{2,2}-$ -solutions of PDE:s	modulus of Brownian Sheet
Chapter 4:	**Averaging Principle:**	**Diffusion Limit:**
inter-arrival time scaling	$\mathcal{O}(\frac{1}{n})$	$\mathcal{O}(\frac{1}{n})$
state scaling	$\mathcal{O}(\frac{1}{n})$	$\mathcal{O}(\frac{1}{\sqrt{n}})$
convergence	almost sure	in distribution
state space, E	p-uniformly smooth Banach space, $p \in (1, 2]$	Hilbert space
limit object	solution of ODE on E	Brownian motion on E

Table 1.1: Overview of the scaling and limits.

2 Laws of Large Numbers for Limit Order Books

In the first section of this chapter, we define a sequence of LOB models indexed by n, where the model with $n = 1$ corresponds to an observable limit order book. We proceed to define the scaling and the main result i.e. a strong law of large numbers. Section 2.2 entails the proof of the main result: in Subsection 2.2.1 we establish convergence of the bid/ask price dynamics to a 2-dimensional ODE and Subsection 2.2.2 is devoted to the analysis of the limiting volume dynamics and its convergence to the solution of a hyperbolic PDE.

In Section 2.3 we show a weak law of large numbers, that is based on an alternative scaling which may be advantageous when proving weak convergence.

2.1 A Strong Law of Large Numbers for LOBs

In electronic markets, orders can be submitted for prices that are multiples of the *price tick* $\Delta x^{(n)}$ which is the smallest increment by which the price can move. We assume that the set of price levels at which orders can be submitted equals $\{x_j^{(n)}\}_{j\in\mathbb{Z}}$, where $\mathbb{Z}$ is the set of integers and $x_j^{(n)} := j \cdot \Delta x^{(n)}$ for $j \in \mathbb{Z}$. The *state* of the order book changes due to incoming order flow and cancelations of standing volume. The state after $k \in \mathbb{N}$ such *events* will be described by a random variable $S_k^{(n)}$ taking values in a suitable *state space* E. The k:th event occurs at a random point in time $\tau_k^{(n)}$. The state and time dynamics are defined, respectively, as

$$S_0^{(n)} := s_0^{(n)}, \quad S_{k+1}^{(n)} := S_k^{(n)} + \mathcal{D}_k^{(n)}(S_k^{(n)}) \tag{2.1.1}$$

and

$$\tau_0^{(n)} := 0, \quad \tau_{k+1}^{(n)} := \tau_k^{(n)} + \mathcal{C}_k^{(n)}(S_k^{(n)}) \tag{2.1.2}$$

where $s_0^{(n)} \in E$ is the deterministic initial state, $\mathcal{D}_k^{(n)} : E \to E$ and $\mathcal{C}_k^{(n)} : E \to [0,\infty)$ are random operators that will be specified below.

2.1.1 Model Dynamics

In the sequel we specify the dynamics of our order book model. In particular we specify how different order types change the state of the book. Throughout, all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Our model is defined in relative coordinates i.e. we consider the buy [sell] limit order volumes to be placed at a certain distance from the best bid [ask]. We believe this to be quite natural, as there are (as noted in the Introduction) queueing aspects in an LOB. The further away from the spread a buy [sell] limit order is placed, the longer it will generally take for that order to be filled as it will be matched only when standing at the best bid [ask]. There are also technical advantages with this kind of modeling as the placement distributions when considered in absolute price coordinates, which realistically change when the prices change, are stationary when considered in relative price coordinates.

The initial state

The initial states of the order book are specified in terms of the original discrete model with $n = 1$, which corresponds to observables in an electronic market and where the price tick $\Delta x^{(1)} = \Delta x$ and the price levels $x_j^{(1)} = x_j$. The initial best bid and ask price is given by the pair (B_0, A_0) of best bid and ask prices together with the standing buy and sell limit order volumes $W_{b,0}^j$ and $W_{s,0}^j$ at the price levels x_j. Since the best bid price is the highest price that a buyer is willing to pay $W_{b,0}^j = 0$ for $x_j \geq B_0$ and $W_{s,0}^j = 0$ for $x_j \leq A_0$ because the best ask price is the lowest price a seller is willing to accept (see Figure 2.1).

We identify standing buy and sell limit order volumes with the *relative volume density functions*

$$v_{b,0}^{(1)}(x) := \sum_{j=-\infty}^{\infty} W_{b,0}^j \frac{\mathbb{1}_{[x_j,x_{j+1})}(B_0 - x)}{\Delta x}, \quad v_{s,0}^{(1)}(x) := \sum_{j=-\infty}^{\infty} W_{s,0}^j \frac{\mathbb{1}_{[x_j,x_{j+1})}(A_0 + x)}{\Delta x}.$$

For $x \geq 0$ these step functions describe the standing volume relative to the best bid and ask price, respectively. In order to conveniently model placements of limit orders into the spread, we extend $v_{b,0}^{(1)}$ and $v_{s,0}^{(1)}$ to the negative half-line. The collection of volumes standing at negative distances from the best bid/ask price is referred to as the *shadow book*. The shadow book will undergo the analogous dynamics as the standing volume ("visible book"). At any point in time it specifies the volumes that will be placed into the spread should such an event occur next[1]. From the above, we specify the initial data

[1]One has to specify the volumes placed into the spread somehow. Our choice of shadow books is one such way. The role of the shadow book will be further clarified in the following subsection when we define impact of order arrivals on the state of the book. Its initial state is part of the model; its state will undergo the same dynamics as those of the visible book.

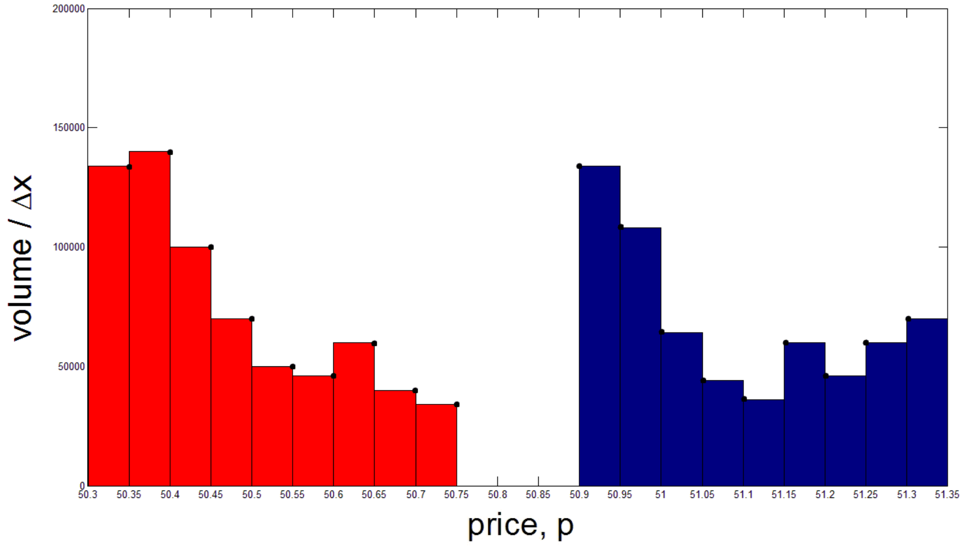

Figure 2.1: A snapshot of the order book in volume density functions and absolute price coordinates. The standing volume at the i:th price level is given by the respective area under the curve between $i\Delta x$ and $(i+1)\Delta x$. In this example the price tick is Δx=0.05.

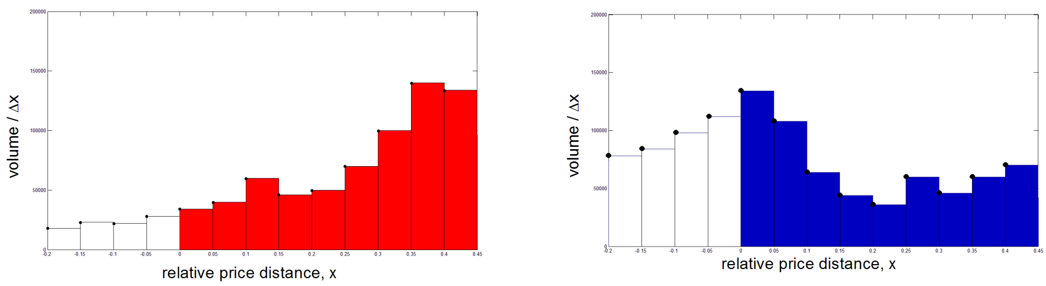

Figure 2.2: Left [right]: the buy [sell] volume density function in relative price coordinates. The colored area under the densities is the standing volume and at negative distances the shadow book models the spread dynamics.

for all $n \geq 1$ in a straightforward way.

Definition 2.1.1. *The initial state of the book is given by a column vector*

$$S_0^{(n)}(\cdot) = \left(B_0, A_0, v_{b,0}^{(n)}(\cdot), v_{s,0}^{(n)}(\cdot)\right)',$$

where $B_0 < A_0$ are the best bid/ask price and the step functions $v_{b,0}^{(n)}, v_{s,0}^{(n)} : \mathbb{R} \to [0, \infty)$ are to be interpreted as follows:

$$v_{b,0}^{(n)}(x) \; \left[v_{s,0}^{(n)}(x)\right] := \begin{cases} \textit{standing buy [sell] limit order volume density at price} \\ \textit{distance } x \textit{ below [above] the best bid [ask] price,} \\ \textit{for } x \geq 0 \textit{ (visible book)} \\ \\ \textit{potential buy [sell] limit order volume density at price} \\ \textit{distance } x \textit{ above [below] the best bid [ask] price,} \\ \textit{for } x < 0 \textit{ (shadow book)} \end{cases} . \tag{2.1.3}$$

More precisely, we assume that the initial standing volumes $v_{b,0}^{(n)}$ and $v_{s,0}^{(n)}$ are given in terms of step functions as follows: for $r \in \{b, s\}$ we put

$$v_{r,0}^{(n)}(x) := \sum_j v_{r,0}^{(n),j} \mathbb{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(x) \quad \textit{and} \quad v_{r,0}^{(n),j} := \frac{1}{\Delta x^{(n)}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} v_{r,0}(x) \mathrm{d}x, \tag{2.1.4}$$

where the functions $v_{r,0}$ $(r \in \{b, s\})$ are assumed to belong to the class $L^2(\mathbb{R}, \mathbb{R}_{>0})$.

Using e.g. an algorithm proposed in Schmidt et. al [69] with added regularity, one can then find a - not necessarily unique - smooth function

$$v_{r,0} \in C_b^2(\mathbb{R}, [0, \infty)) \cap L^2(\mathbb{R}, [0, \infty)), \quad \textit{with} \quad v'_{r,0}, v''_{r,0} \in L^2(\mathbb{R}, \mathbb{R}) \quad \textit{for } r = b, s \tag{2.1.5}$$

that satisfies

$$\frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} v_{b,0}(x) \mathrm{d}x = W_{b,0}^{\frac{B_0}{\Delta x} - j} \qquad \textit{and} \qquad \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} v_{s,0}(x) \mathrm{d}x = W_{s,0}^{\frac{A_0}{\Delta x} + j}. \tag{2.1.6}$$

Defining initial standing volumes as originating from a common "base function" guarantees convergence of initial states in a suitable sense, a necessary condition to establish convergence of the sequence of scaled models to a continuous function limit.

Event types

There are eight events - labelled A, ..., H - that change the state of the book:

$$\begin{aligned} &\mathbf{A} := \{\text{market sell order}\} && \mathbf{B} := \{\text{buy limit order placed in the spread}\} \\ &\mathbf{C} := \{\text{cancelation of buy volume}\} && \mathbf{D} := \{\text{buy limit order not placed in spread}\} \\ &\mathbf{E} := \{\text{market buy order}\} && \mathbf{F} := \{\text{sell limit order placed in the spread}\} \\ &\mathbf{G} := \{\text{cancelation of sell volume}\} && \mathbf{H} := \{\text{sell limit order not placed in the spread}\}. \end{aligned}$$

We assume that no two events happen simultaneously and describe the state dynamics with a stochastic process $\{S_k^{(n)}\}_{k\in\mathbb{N}}$, where $S_k^{(n)}(\cdot) := \left(B_k^{(n)}, A_k^{(n)}, v_{b,k}^{(n)}(\cdot), v_{s,k}^{(n)}(\cdot)\right)'$ that takes values in the Hilbert space

$$E := \mathbb{R} \times \mathbb{R} \times L^2(\mathbb{R},\mathbb{R}) \times L^2(\mathbb{R},\mathbb{R}).$$

The first two components of the vector $S_k^{(n)}$ stand for the best bid and ask price after k events; the third and fourth component refer to the relative buy and sell volume density functions, respectively (visible and shadow book), in the n:th model respectively. We define a norm on E by

$$\|\alpha\|_E := |\alpha_1| + |\alpha_2| + \|\alpha_3\|_{L^2} + \|\alpha_4\|_{L^2}, \quad \alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in E. \tag{2.1.7}$$

We are now going to specify how, in our model, the different events change the state of the book.

Active orders

Market orders and placements of limit orders in the spread lead to price changes[2]. With a slight abuse of terminology we refer to these order types as *active orders*. For convenience we assume that market orders match precisely against the standing volume at the best bid price and that all buy limit orders placed in the spread improve prices by one tick. The analogous condition is assumed for the sell side.

If the k:th event is a market sell order (Event **A**), then the relative buy volume density shifts one price tick to the left, the best bid price decreases by one tick and the best ask price remains unchanged. One has that

$$B_{k+1}^{(n)} = B_k^{(n)} - \Delta x^{(n)}, \quad A_{k+1}^{(n)} = A_k^{(n)}.$$

[2] A market order that does not lead to a price change can be viewed as a cancelation of standing volume at the best bid/ask price.

To describe the dynamics of the volumes, which shift when active orders arrive, we introduce a *translation operator*. It can easily be shown to be linear, commutative and isometric, see Lemma 5.2.4 in the Appendix.

Definition 2.1.2 (Translation operator)**.** *The translation operators*

$$T_+^{(n)}, T_-^{(n)} : L^2(\mathbb{R}, \mathbb{R}) \to L^2(\mathbb{R}, \mathbb{R})$$

shift relative volume densities $v(\cdot) \in L^2(\mathbb{R}, \mathbb{R})$ one price tick $\Delta x^{(n)}$ to the left and right respectively:

$$T_+^{(n)}\Big(v(\cdot)\Big) = v(\cdot + \Delta x^{(n)}), \quad T_-^{(n)}\Big(v(\cdot)\Big) = v(\cdot - \Delta x^{(n)}).$$

The placement of orders into the spread will be modeled using the shadow book. If the k:th event is a buy limit order placement in the spread (Event **B**), the relative buy volume density shifts one price tick to the right, the best bid price increases by one tick and the relative sell volume density and the best ask price remain unchanged:

$$v_{b,k+1}^{(n)}(\cdot) = T_-^{(n)}\left(v_{b,k}\right)(\cdot), \quad v_{s,k+1}^{(n)}(\cdot) = v_{s,k}^{(n)}(\cdot)$$

and

$$B_{k+1}^{(n)} = B_k^{(n)} + \Delta x^{(n)}, \quad A_{k+1}^{(n)} = A_k^{(n)}.$$

Remark 2.1.3. *Notice that market order arrivals and limit order placements in the spread are "inverse operations": a market sell order arrival followed by a limit buy order placement in the spread (or vice versa) leaves the book unchanged.*

Passive orders

Limit order placements outside the spread and cancelations of standing volume do not change prices. We refer to these order types as *passive orders*. Limit buy order placements outside the spread (Event **D**) occur for a random *volume* $\omega_k^{(n),D}$ at a random price distance π_k^D from the best bid price. Thus, if the k:th event is a limit buy order arrival, then

$$v_{b,k+1}^{(n)}(\cdot) = v_{b,k}^{(n)}(\cdot) + M_k^{(n),D}(\cdot), \quad \text{where} \quad M_{v,k}^{(n),D}(x) := \omega_k^{(n),D} \sum_{j=-\infty}^{\infty} \frac{\mathbb{1}_{\{x, \pi_k^D \in [x_j^{(n)}, x_{j+1}^{(n)})\}}}{\Delta x^{(n)}}.$$

Cancelations of standing buy volume (Event **C**) occur at random price levels π_k^C for random *proportions* $\omega_k^{(n),C} \in (0, \frac{\Delta x^{(n)}}{f(n)})$ of the standing volume density, which corresponds to a proportion between 0 and 1 of the standing volume over the price level. The scaling function f will be specified in Assumption 2.1.6 below.

Thus, if the k:th event is a cancelation, then

$$v_{b,k+1}^{(n)}(\cdot) = v_{b,k}^{(n)}(\cdot) - M_k^{(n),C}(\cdot)v_{b,k}(\cdot), \text{ where } M_{v,k}^{(n),C}(x) := \omega_k^{(n),C} \sum_{j=-\infty}^{\infty} \frac{\mathbb{1}_{\{x,\pi_k^C \in [x_j^{(n)}, x_{j+1}^{(n)})\}}}{\Delta x^{(n)}}$$

and in either case, the bid/ask price and standing sell limit order volume remain unchanged:

$$v_{s,k+1}^{(n)}(\cdot) = v_{s,k}^{(n)}(\cdot), \quad B_{k+1}^{(n)} = B_k^{(n)}, \quad A_{k+1}^{(n)} = A_k^{(n)}.$$

Similar considerations apply to the sell side with respective random quantities $\omega_k^{(n),G}$, $\omega_k^{(n),H}$ and π_k^G, π_k^H. Since buy (sell) orders are placed/canceled at a random distance from the best ask (bid) price, changes in the standing volume take place either in the visible or the shadow book, depending on whether the realizations of the random variables π_k^I are positive or negative; proportional cancelations guarantee non-negativity of standing volumes.

Remark 2.1.4. *The placement/cancelation events are scaled via the sequence of tick intervals $\left\{[x_j^{(n)}, x_{j+1}^{(n)})\right\}_{j\in\mathbb{Z},n\in\mathbb{N}}$. Thus, the random price distances themselves do not scale in n i.e. $\pi_k^{(n),I} = \pi_k^I$ for all I, k and n. Choosing the price tick $\Delta x^{(n)}$ as our unit for relative volume cancelations allows us to use the same convenient form for the scaling of limit order placement and cancelation dynamics.*

Event dynamics

In order to specify the dynamics of the events, we associate with each order type $I \in \{A, ..., H\}$ a sequence of inter arrival times $\{\phi_k^{(n),I}\}_{k\geq 0, n\geq 1}$ and since we will assume that no two events happen simultaneously, we introduce the k:th *event indicators* that specifies the type of the k:th event in terms of the inter arrival times:

$$\mathbb{1}_k^{(n),I}\left(S_k^{(n)}\right) := \begin{cases} 1, & \text{if } \phi_k^{(n),I}(S_k^{(n)}) = \min\{\phi_k^{(n),A}(S_k^{(n)}), \dots, \phi_k^{(n),H}(S_k^{(n)})\} \\ 0, & \text{otherwise} \end{cases} \tag{2.1.8}$$

for $I \in \{A, \dots, H\}$ and we denote the 8×1 event indicator vector

$$\mathbb{1}_k^{(n)}\left(S_k^{(n)}\right) := \left(\mathbb{1}_k^{(n),A}\left(S_k^{(n)}\right), \dots, \mathbb{1}_k^{(n),H}\left(S_k^{(n)}\right)\right)'. \tag{2.1.9}$$

The random event times $\tau_k^{(n)}$ are defined recursively as

$$\tau_0^{(n)} := 0, \quad \tau_{k+1}^{(n)} := \tau_k^{(n)} + C_k^{(n)}(S_k^{(n)}) \tag{2.1.10}$$

where the random inter arrival time operator $C_k^{(n)} : E \to \mathbb{R}_+$ is given by

$$C_k^{(n)}(\cdot) := \min_{I \in \{A,\ldots,H\}} \Phi_k^{(n),I}(\cdot) \quad \text{for the inter arrival times } \Phi_k^{(n),I}(\cdot).$$

The state dynamics of the order book models are given by

$$S_{k+1}^{(n)} := S_k^{(n)} + \mathcal{D}_k^{(n)}(S_k^{(n)}), \tag{2.1.11}$$

where the 4×1 random state operator $\mathcal{D}_k^{(n)}$ is defined by

$$\mathcal{D}_k^{(n)}(S_k^{(n)}) := \mathbb{M}_k^{(n)}(S_k^{(n)}) \cdot \mathbb{1}_k^{(n)}\left(S_k^{(n)}\right), \tag{2.1.12}$$

where the 4×8 matrix $\mathbb{M}_k^{(n)}(S_k^{(n)}) := \left(\mathbb{M}_{buy,k}^{(n)}(S_k^{(n)}), \mathbb{M}_{sell,k}^{(n)}(S_k^{(n)})\right)$ consists of the buy side dynamics

$$\mathbb{M}_{buy,k}^{(n)}(S_k^{(n)}) := \begin{pmatrix} -\Delta x^{(n)} & \Delta x^{(n)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{v,k}^{(n),A}(v_{b,k}^{(n)}) & M_{v,k}^{(n),B}(v_{b,k}^{(n)}) & -M_{v,k}^{(n),C} v_{b,k}^{(n)} & M_{v,k}^{(n),D} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the sell side dynamics

$$\mathbb{M}_{sell,k}^{(n)}(S_k^{(n)}) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \Delta x^{(n)} & -\Delta x^{(n)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{v,k}^{(n),E}(v_{s,k}^{(n)}) & M_{v,k}^{(n),F}(v_{s,k}^{(n)}) & -M_{v,k}^{(n),G} v_{s,k}^{(n)} & M_{v,k}^{(n),H} \end{pmatrix}.$$

The entries $M_{v,k}^{(n),I}$ for price changing events are given by

$$M_{v,k}^{(n),I}(v_{r,k}^{(n)}) := T_+^{(n)}\left(v_{r,k}^{(n)}\right) - v_{r,k}^{(n)}, \quad \text{for pairs } (r,I) \in \{(b,A),(s,E)\} \tag{2.1.13}$$

and

$$M_{v,k}^{(n),I}(v_{r,k}^{(n)}) := T_-^{(n)}\left(v_{r,k}^{(n)}\right) - v_{r,k}^{(n)}, \quad \text{for pairs } (r,I) \in \{(b,B),(s,F)\}. \tag{2.1.14}$$

We have that the random one step function

$$M_{v,k}^{(n),I}(x) := \omega_k^{(n),I} \sum_{j=-\infty}^{\infty} \frac{\mathbb{1}_{\{x,\pi_k^I \in [x_j^{(n)}, x_{j+1}^{(n)})\}}}{\Delta x^{(n)}}, \quad I \in \{C, D, G, H\} \tag{2.1.15}$$

defines the scaled fluctuations of the limit order placements and cancelations.

Remark 2.1.5. *If e.g. the inter arrival time* $\phi_k^{(n),C} = \min\{\phi_k^{(n),A}, \ldots, \phi_k^{(n),H}\}$ *i.e. the Event* $\boldsymbol{C}$*, a buy limit order cancelation occurs, then* $\mathbb{1}_k^{(n)}(S_k^{(n)}) = (0,0,1,0,0,0,0,0)'$ *and the dynamics of the random operator* $\mathcal{D}_k^{(n)}(S_k^{(n)})$ *will be those of the third column of the matrix* $\mathbb{M}_k(S_k^{(n)})$*.*

When observing the model process in continuous time, we have

$$S^{(n)}(t) := S_k^{(n)} \quad \text{as} \quad t \in [\tau_k^{(n)}, \tau_{k+1}^{(n)}), \quad t \geq 0. \tag{2.1.16}$$

It follows that the paths of the stochastic process $S^{(n)}(t)$ are càdlàg, i.e. are right continuous and have left limits and we write $S^{(n)}(t)(\omega) \in D([0,\infty), E)$.

Our convergence results will be for $t \in [0,T]$, and thus we have

$$S^{(n)}(\cdot) : (\Omega, \mathcal{F}, \mathbb{P}) \to \Big(D([0,T], E), \mathcal{B}(D([0,T], E))\Big).$$

When the limiting process is continuous in the time parameter (as it is in our case) we may endow $D([0,T],E)$ with the uniform topology, with respect to the norm (see Billingsley [12, p.124]):

$$\|f\|_T := \sup_{t\in[0,T]} \|f(t)\|_E \quad \text{for } f \in D([0,T], E). \tag{2.1.17}$$

2.1.2 Scaling Assumptions

Our goal is to associate a scaling limit for the sequence of LOB models in such a way that the limiting dynamics are defined in terms of a coupled PDE:ODE system. To this end, we will assume that there are two time scales[3] in the n:th model:

[3] Empirical studies have implied that there indeed are different time and volume scales in an LOB. E.g. Hautsch and Huang [42, p.3] state: "A detailed analysis of the NASDAQ order flow in October 2010 provides the following major results: First, the number of limit order submissions is twenty to forty higher than the number of trades. Secondly, limit order sizes are typically small and clustered at round lot sizes of hundred shares. Third, more than 95% of all limit orders are canceled without getting executed with most of them being canceled nearly instantaneously (less than one second) after their submission reflecting the proliferation of algorithmic trading at NASDAQ. Fourth, volume-weighted execution times are significantly greater than average execution times indicating that large orders face more execution risk than small ones."

- a *slow* time scale for active orders (events A,B,E and F) with inter arrival times and price movements of order $\mathcal{O}\left(\frac{1}{n}\right)$
- a *fast* time scale for passive orders (events C,D,G and H) with inter arrival times and placement/cancelation volumes of order $\mathcal{O}\left(\frac{1}{n^s}\right)$.

We also assume that the coupling of the best bid and ask side occurs over the random inter arrival times, which are assumed to be functions of the best bid and ask price. The details and technical conditions of our assumptions are given below.

Assumption 2.1.6. *Let $s := 1 + \frac{2}{\delta}$ and $\delta \in (0, 1)$. We assume that*

- *The price tick scales by a factor $\frac{1}{n}$ and the time tick by a factor of $\frac{1}{n^s}$:*

$$\Delta x^{(n)} := \frac{\Delta x}{n} \quad \text{and} \quad \Delta t^{(n)} := \frac{\Delta t}{n^s}. \tag{2.1.18}$$

- *The placement and cancelation volumes scale by a factor of $\frac{1}{n^s}$:*

$$\omega_k^{(n),I} := \frac{\omega_k^I}{n^s}, \quad I \in \{A, \ldots, H\}. \tag{2.1.19}$$

- *The inter arrival times for active orders and passive orders scale by a factor of $\frac{1}{n}$ and $\frac{1}{n^s}$, respectively:*

$$\Phi_k^{(n),I}(\cdot) := \begin{cases} \frac{\Phi_k^I(\cdot)}{n}, & I \in \{A, B, E, F\} \\ \\ \frac{\Phi_k^I(\cdot)}{n^s}, & I \in \{C, D, G, H\} \end{cases}. \tag{2.1.20}$$

In terms of our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we have that $\mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}$, where we set the sequence of σ-algebras to be generated by the unscaled initial conditions $s_0^{(1)} := \left(B_0, A_0, v_{b,0}^{(1)}(\cdot), v_{s,0}^{(1)}(\cdot)\right)'$ and the random order flow i.e.

$$\mathcal{F}_{-1} := \sigma(s_0^{(1)}) \tag{2.1.21}$$

and

$$\mathcal{F}_k := \sigma\left(s_0^{(1)}, \left\{(\pi_i^I, \omega_i^I, \Phi_i(\alpha)), I \in \{A, \ldots, H\}, \alpha \in \mathbb{R}^2\right\}_{i=0}^k\right), \quad k \geq 0. \tag{2.1.22}$$

We assume that given the current bid and ask price, the random variables $(\pi_k^I, \omega_k^I, \Phi_k(\alpha))$ are jointly independent families with finite second moments and each of them identically distributed for each $k \geq 0$.

The inter arrival times $\phi_k^I : \Omega \times \mathbb{R}^2 \to [0,\infty)$ *are assumed to be conditionally independent, given the current best bid and ask price with continuously differentiable conditional cumulative distribution functions*

$$H_{\phi^I}(t) := \mathbb{P}\left(\phi_k^I(\cdot,\cdot) \leq t \middle| S_k^{(n)}\right) = \mathbb{P}\left(\phi_k^I(\cdot,\cdot) \leq t \middle| B_k^{(n)}, A_k^{(n)}\right).$$

Due to this assumption and the dynamics of the best bid and ask price, it is purposeful to define the sequence of event sub-σ-algebras $\mathcal{F}_{-1}^{\Phi} \subset \mathcal{F}_0^{\Phi} \subset \mathcal{F}_1^{\Phi} \subset \ldots \subset \mathcal{F}^{\Phi} \subset \mathcal{F}$ *:*

$$\mathcal{F}_{-1}^{\Phi} := \sigma\left(B_0, A_0\right) \tag{2.1.23}$$

and

$$\mathcal{F}_k^{\Phi} := \sigma\left(B_0, A_0, \left\{\Phi_i(\alpha), I \in \{A,\ldots,H\}, \alpha \in \mathbb{R}^2\right\}_{i=0}^{k}\right), \quad k \geq 0. \tag{2.1.24}$$

It follows by (2.1.8) and (2.1.11) that in the n:th model, the probability of the next event being of type $I \in \{A,\ldots,H\}$ *is given by*

$$\begin{aligned}\mathbb{P}\Bigg(\Phi_k^{(n),I}\left(B_k^{(n)}, A_k^{(n)}\right) = \min\left\{\Phi_k^{(n),A}\left(B_k^{(n)}, A_k^{(n)}\right), \ldots, \Phi_k^{(n),H}\left(B_k^{(n)}, A_k^{(n)}\right)\right\} \Big| \mathcal{F}_{k-1}^{\Phi}\Bigg)\\ = \mathbb{E}\left[\mathbb{1}_k^{(n),I}\left(B_k^{(n)}, A_k^{(n)}\right) \middle| \mathcal{F}_{k-1}^{\Phi}\right]\end{aligned}$$

and we assume that for $\mathcal{F}_{k-1}^{\Phi}$*-measurable arguments, such as the best bid and ask prices* $\left(B_k^{(n)}, A_k^{(n)}\right)$*, which we denote* $(\cdot,\cdot)$ *that*

$$\mathbb{E}\left[\mathbb{1}_k^{(n),I}(\cdot,\cdot) \middle| \mathcal{F}_{k-1}^{\Phi}\right] = \begin{cases} \frac{p^{(n),I}(\cdot,\cdot)}{n^{s-1}}, & \text{for } I \in \{A,B,E,F\} \\ \\ p^{(n),I}(\cdot,\cdot), & \text{for } I \in \{C,D,G,H\} \end{cases} \tag{2.1.25}$$

and that the moments of the inter arrival time operator satisfy

$$\mathbb{E}\left[\mathcal{C}_k^{(n)}(\cdot,\cdot) \middle| \mathcal{F}_{k-1}^{\Phi}\right] = m^{(n)}(\cdot,\cdot) \cdot \Delta t^{(n)}$$

$$\mathbb{E}\left[\left(\mathcal{C}_k^{(n)}(\cdot,\cdot)\right)^2 \middle| \mathcal{F}_{k-1}^{\Phi}\right] = r^{(n)}(\cdot,\cdot) \cdot \left(\Delta t^{(n)}\right)^2 \tag{2.1.26}$$

such that

$$p^{(n),I}(\cdot,\cdot) \to p^{*,I}(\cdot,\cdot),\; m^{(n)}(\cdot,\cdot) \to m^{*}(\cdot,\cdot) \quad \text{and} \quad r^{(n)}(\cdot,\cdot) \to r^{*}(\cdot,\cdot), \tag{2.1.27}$$

uniformly as $n \to \infty$*, where* $p^{(n),I}, p^{*,I} \in C^{1,1}(\mathbb{R}\times\mathbb{R},[0,1))$*,* $m^{(n)}, m^{*,I}, r^{(n)}, r^{*,I} \in$

$C^{1,1}(\mathbb{R} \times \mathbb{R}, \mathbb{R}_+)$ and all these functions have bounded first derivatives.

Remark 2.1.7. *The scaled volume (2.1.19), inter arrival times (2.1.20) and the placement/cancelation function (2.1.15) are measurable functions of the random variables ω_k^I, ϕ_k^I and π_k^I for all n, since these are $\mathcal{F}_k$-measurable where $\mathcal{F}_k$ is given by (2.1.22). From the recurrent dynamics definitions (2.1.10)-(2.1.11), it follows that the states $S_k^{(n)}$ and random times $\tau_k^{(n)}$ are $\mathcal{F}_{k-1}$-measurable. Furthermore, the random times $\tau_k^{(n)}$ and the best bid and ask prices $B_k^{(n)}$ and $A_k^{(n)}$ are measurable w.r.t. the event sub-σ-algebra $\mathcal{F}_{k-1}^{\Phi}$, while the buy and sell volume densities $v_{b,k}^{(n)}$ and $v_{s,k}^{(n)}$ are not, as the random volumes and prices (ω_k^I and π_k^I) are independent of this sub-σ-algebra.*

That the dynamics of the best and bid ask price drive the dynamics of the order book, is given by the state-dependence of the inter arrival times. Thus, the dynamics of the sequence of event times $\{\tau_k^{(n)}\}$ in (2.1.10) satisfy

$$\tau_{k+1}^{(n)} = \tau_k^{(n)} + \mathcal{C}_k^{(n)}(B_k^{(n)}, A_k^{(n)}), \text{ where}$$

$$\mathcal{C}_k^{(n)}(\cdot,\cdot) = \min\left\{\phi_k^{(n),A}(\cdot,\cdot), \ldots, \phi_k^{(n),H}(\cdot,\cdot)\right\}.$$

We notice that the prevailing best bid and ask prices are a sufficient statistic for distribution of the *change* of the order book, by our choice of modeling. In particular, the dynamics on the level of prices does not depend on the liquidity available for trading. It is this simplifying assumption that will allow us to eventually establish a PDE scaling limit for the order book dynamics.

We assume that placements and cancelations of limit buy (sell) orders occur at random price levels relative to the best ask (bid) price independently of the state of volume densities, and that no placements take place at a distance larger than M for some $M > 0$. More precisely, we assume that the following technical conditions are satisfied:

Assumption 2.1.8. *For the placement and cancelation prices π_k^I, $I \in \{C, D, G, H\}$, there exist probability density functions,*

$$f^I \in \begin{cases} C^2([-M, M], [0, \infty)) & I \in \{D, H\} \\ \\ C^2([-M, M], [0, 1)), & I \in \{C, G\}, \end{cases} \tag{2.1.28}$$

where $f^I(x) = 0, \quad x \in \mathbb{R} \setminus (-M, M)$, $I \in \{C, D, G, H\}$.

Assumption 2.1.8 allows us to conveniently specify the corresponding expected placement/cancelation functions in the scaled models in terms of the same functions f^I in analogous fashion as the initial volume (2.1.4) above.

It holds that

$$\mathbb{E}\left[M_{v,k}^{(n),I}(x)\right] = \mu_{\omega^{(n),I}} f^{(n),I}(x), \quad \text{where} \quad \mu_{\omega^{(n),I}} := \mathbb{E}[\omega_k^{(n),I}],$$

$$f^{(n),I} = \sum_{j=-\infty}^{\infty} f^{(n),I,j} \mathbb{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(x)$$

and

$$f^{(n),I,j} = \frac{\mathbb{P}\left(\left\{\pi_k^I \in [x_j^{(n)}, x_{j+1}^{(n)})\right\}\right)}{\Delta x^{(n)}} = \frac{1}{\Delta x^{(n)}} \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} f^I(x)\mathrm{d}x. \tag{2.1.29}$$

2.1.3 The Main Result

The main result of the first part is Theorem 2.1.9, which states that in relative price coordinates the scaled discrete models converge to a macroscopic limit model, in the sense of a strong law of large numbers. In the limit model, the best bid and ask prices are given as the unique $C^2([0,T],\mathbb{R})$ solution of a first-order autonomous system of two coupled ODE:s. The relative buy and sell volume densities of the continuous model are the respective unique $C^{2,2}((-\infty,\infty)\times[0,T],\mathbb{R}_{>0})$ solution of a first order linear hyperbolic PDE with variable coefficients. A simple transformation yields the dynamics of the limit model in the observed price-volume coordinates and this is expressed in Corollary 2.1.10. We conclude the section with a worked out example for a two-sided state-dependent order book in which order arrivals are modeled as conditionally independent Poisson processes.

Theorem 2.1.9 (Strong Law of Large Numbers for limit order books)**.** *Let $\{S^{(n)}\}_{n\geq 1}$ be a sequence of order book models defined by (2.1.10)-(2.1.11) and suppose that Assumptions 2.1.6 and 2.1.8 hold.*

Then, for all $T>0$ we have that

$$\sup_{t\in[0,T]} \|S^{(n)}(\cdot,t) - s(\cdot,t)\|_E \to 0, \quad \textit{a.s. as } n\to\infty,$$

where

$$s(\cdot,t) = \begin{pmatrix} \gamma(t) \\ v(\cdot,t) \end{pmatrix},$$

$\gamma(t) = \begin{pmatrix} b(t) \\ a(t) \end{pmatrix}$ *denotes the best bid and ask price, where* $b, a \in C^2([0,T], \mathbb{R})$ *solve*

$$\begin{cases} \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = \frac{A(\gamma(t))}{m^*(\gamma(t))} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in [0,T] \\ \\ \gamma(0) = \begin{pmatrix} B_0 \\ A_0 \end{pmatrix} \end{cases} \tag{2.1.30}$$

uniquely and $v(x,t) = \begin{pmatrix} v_b(x,t) \\ v_s(x,t) \end{pmatrix}$ *denotes the relative buy and sell volume densities, where*

$v_b, v_s \in C^{2,2}(\mathbb{R} \times [0,T], \mathbb{R}_{>0})$ *are the unique solutions of the PDE:s for* $(x,t) \in \mathbb{R} \times [0,T]$*:*

$$\begin{cases} v_t(x,t) = \frac{1}{m^*(\gamma(t))} \Big(A\left(\gamma(t)\right) v_x(x,t) + B(x, \gamma(t)) v(x,t) + c\left(x, \gamma(t)\right) \Big) \\ \\ v(x,0) = v_0(x) \end{cases} \tag{2.1.31}$$

on the relative price interval and with variable coefficients

$$A(\cdot) := \frac{\Delta x}{\Delta t} \begin{pmatrix} p^{*,A}(\cdot) - p^{*,B}(\cdot) & 0 \\ 0 & p^{*,E}(\cdot) - p^{*,F}(\cdot) \end{pmatrix}, \tag{2.1.32}$$

$$B(x, \cdot) := -\frac{1}{\Delta t} \begin{pmatrix} \mu_{\omega^C} p^{*,C}(\cdot) f^C(x) & 0 \\ 0 & \mu_{\omega^G} p^{*,G}(\cdot) f^G(x) \end{pmatrix} \tag{2.1.33}$$

$$c(x, \cdot) := \frac{1}{\Delta t} \begin{pmatrix} \mu_{\omega^D} p^{*,D}(\cdot) f^D(x) \\ \\ \mu_{\omega^H} p^{*,H}(\cdot) f^H(x) \end{pmatrix} \tag{2.1.34}$$

where Δx *and* Δt *are the original price and time tick, respectively.* $p^{*,I}$, $I \in \{A, \ldots, H\}$ *and* $m^*(\cdot)$ *are the limiting probability and time tick functions in (2.1.27).* $\mu_{\omega^I} = \mathbb{E}[\omega_k^I]$ *is the expected placement volume/cancelation proportion and* f^I *the density of the placement price* π^I*.*

Returning to absolute price and time coordinates, we get a convenient expression for both sides of the order book in absolute coordinates, via the definition of the relative volume densities (2.1.3):

Corollary 2.1.10 (Time evolution of standing limit order volume)**.** *For the limit model, the observable standing limit order density at time t is given by*

$$w(p,t) := v_b\left(b(t)-p,t\right)\mathbb{1}_{[0,b(t))}(p) + v_s\left(p-a(t),t\right)\mathbb{1}_{[a(t),\infty)}(p),$$

where p denotes the price of the security, the best bid and ask prices $b,a \in C^2([0,T],\mathbb{R})$ solve (2.1.30) uniquely, the relative buy and sell volume densities $v_b, v_s \in C^{2,2}(\mathbb{R}\times[0,T],\mathbb{R}_{>0})$ solve (2.1.31) uniquely.

2.1.4 Example: Random Order Flow with Poisson Arrivals

In many LOB models, e.g. those in Cont et. al. [17] and Luckock [57], one assumes that orders arrive according to independent Poisson processes. Seen as renewal processes, the inter arrival times of these processes are exponentially distributed i.e. the jump times live on the positive real line.

We consider an order book model in which the event arrivals for the events $A,\dots,H$ are given by conditionally independent Poisson processes with the rates $\lambda^A(b,a)$,...,$\lambda^H(b,a)$ that are functions of the current best bid and ask prices b and a.

In particular, let the best bid/ask-dependent arrival rates for the events have uniformly bounded first derivatives, i.e. $\lambda^I \in C_b^{1,1}(\mathbb{R}\times\mathbb{R},\mathbb{R})$, for $I\in\{A,\dots,H\}$. We now consider a sequence of such models and scale the arrival rates as follows

$$\lambda^{(n),I}(b,a) := \begin{cases} n\lambda^I(b,a), & I = A,B,E,F \\ \\ n^s\lambda^I(b,a), & I = C,D,G,H \end{cases}$$

i.e. the arrival rates of price changing orders are scaled by a factor of n (analogous to the scaling of state dependent Poisson processes in Mandelbaum et. al. [58]) and the arrival rates of volume fluctuations by a factor of n^s. We have from (2.1.25) of Assumption 2.1.6 that

$$\begin{aligned}&\mathbb{P}\left(\{\text{event } I \text{ happens next, given the current best bid and ask prices } b \text{ and } a\ \}\right)\\ &= \mathbb{E}\left[\mathbb{1}_k^{(n),I}(b,a)\,\middle|\mathcal{F}_{k-1}^{\Phi}\right] = \frac{\lambda^{(n),I}(b,a)}{\lambda^{(n),A}(b,a)+\ldots+\lambda^{(n),H}(b,a)}\end{aligned}$$

and the conditional expected inter arrival time is given by

$$\begin{aligned}&\mathbb{E}\left[C_k^{(n)}(b,a)\,\middle|\mathcal{F}_{k-1}^{\Phi}\right]\\ &= \mathbb{E}\left[\min\left\{\Phi_k^{(n),A}(b,a),\dots,\Phi_k^{(n),H}(b,a)\right\}\middle|\mathcal{F}_{k-1}^{\Phi}\right] = \frac{1}{\lambda^{(n),A}(b,a)+\ldots+\lambda^{(n),H}(b,a)}.\end{aligned}$$

This implies that relations (2.1.25)-(2.1.27) of Assumption 2.1.6 hold with

$$p^{*,I}(\cdot,\cdot) = \frac{\lambda^I(\cdot,\cdot)}{\lambda^C(\cdot,\cdot)+\lambda^D(\cdot,\cdot)+\lambda^G(\cdot,\cdot)+\lambda^H(\cdot,\cdot)}, \quad I = A,\dots,H \quad \text{and}$$

$$m^*(\cdot,\cdot) = \frac{1}{\Delta t}\frac{1}{\lambda^C(\cdot,\cdot)+\lambda^D(\cdot,\cdot)+\lambda^G(\cdot,\cdot)+\lambda^H(\cdot,\cdot)}.$$

Let the remaining order flow parameters scale according to Assumptions 2.1.6 and 2.1.8. Then by, our main result Theorem 2.1.9, for all $T > 0$ and $t \in [0,T]$ the best bid and ask prices in the limit model solve

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} b \\ a \end{pmatrix} = \Delta x \begin{pmatrix} \lambda^B(b,a) - \lambda^A(b,a) \\ \lambda^E(b,a) - \lambda^F(b,a) \end{pmatrix}$$

and the sell [buy] volume density, see (2.2.47)-(2.2.48), is given by the general solution[4]

$$v_s(x,t) = \bar{v}_s(x + F_s(t), t), \quad \text{where}$$

$$\bar{v}_s(\xi,t) = \exp\left(\int_0^t g_s(\xi - F_s(u), u)\mathrm{d}u\right) \cdot v_{s,0}(\xi), \quad \text{[m.m.for the buy side]}$$

$$F_s(t) = \Delta x \int_0^t \left\{\lambda^E(b(u),a(u)) - \lambda^F(b(u),a(u))\right\} \mathrm{d}u \quad \text{and}$$

$$g_s(x,s) = \lambda^H(b,a)\mu_{\omega^H} f^H(x) - \lambda^G(b,a)\mu_{\omega^G} f^G(x).$$

By Corollary 2.1.10, the observable standing limit order volume density is given by

$$w(p,t) := v_b\left(b(t) - p, t\right) \mathbb{1}_{[0,b(t))}(p) + v_s\left(p - a(t), t\right) \mathbb{1}_{[a(t),\infty)}(p),$$

where p is the price of the security.

[4]See (2.2.49) in the proof of Proposition 2.2.6.

2.2 Proof of the Strong Law of Large Numbers

The proof of Theorem 2.1.9 relies on expressing the model sequence $S^{(n)}$ as the composition of its state process $\eta^{(n)}$ (with the random values of the best bid [ask] price and the relative buy [sell] limit volume densities) and a time process $\mu^{(n)}$ (see Proposition 4.2.6):

$$S^{(n)}(t) = \eta^{(n)}\left(\mu^{(n)}(t) - \Delta t^{(n)}\right).$$

The advantage of this representation is that the processes $\eta^{(n)}$ and $\mu^{(n)}$ jump at deterministic equidistant points in time $t_k^{(n)} := k\Delta t^{(n)}$ for $k = 0, \ldots, \lfloor \frac{t}{\Delta t^{(n)}} \rfloor$, by the natural càdlàg partition of the time interval[5]. We then show the convergence of the state and the time process

$$\sup_{t\in[0,T)} \left\|\eta^{(n)}(t) - \eta(t)\right\|_E \to 0 \quad a.s. \quad \text{and} \quad \sup_{t\in[0,T]} \left|\mu^{(n)}(t) - \mu(t)\right| \to 0 \quad a.s. \quad \text{as } n \to \infty,$$

to the processes η and $\mu(t) = y^{-1}(t)$, where η and y solve ODE:s on our state space E and time interval $[0, T]$, respectively.

By the Time Change Theorem 4.2.7 it then holds that

$$S^{(n)}(t) = \eta^{(n)}\left(\mu^{(n)}(t) - \Delta t^{(n)}\right) \to \eta(\mu(t)) =: s(t) \quad \text{a.s.} \tag{2.2.1}$$

and uniformly for $t \in [0, T]$ as $n \to \infty$.

A fundamental part of the proof is to identify the limiting ODE:s for the time and state process, respectively:

$$\begin{cases} \frac{\mathrm{d}\eta}{\mathrm{d}t}(t) &= b^*(\eta(t)), \quad t \in (0, T] \\ \\ \eta(0) &= s_0 \end{cases} \tag{2.2.2}$$

and

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t}(t) &= m^*(\eta(t)), \quad t \in (0, T] \\ \\ y(0) &= 0. \end{cases} \tag{2.2.3}$$

Thus, to specify the equation for the limiting order book model one needs to find the deterministic operators b^* and m^*. We get, using the above equations (2.2.1)-(2.2.3) and applying the chain rule for functions on Banach spaces (see e.g. Jost [49, Theorem

[5]One has $[0, t) := \bigcup_{k=0}^{\lfloor \frac{t}{\Delta t^{(n)}} \rfloor} \left[t_k^{(n)}, t_{k+1}^{(n)}\right)$.

8.4 on p.105]), that for the limiting order book process $s(t)$:

$$\begin{aligned}\frac{\mathrm{d}s(t)}{\mathrm{d}t} &= \frac{\mathrm{d}\eta\left(\mu(t)\right)}{\mathrm{d}t} = \frac{\mathrm{d}\eta}{\mathrm{d}t}\left(\mu(t)\right)\cdot\frac{\mathrm{d}\mu}{\mathrm{d}t}(t) = \frac{\mathrm{d}\eta}{\mathrm{d}t}\left(\mu(t)\right)\cdot\frac{\mathrm{d}\left(y^{-1}\right)}{\mathrm{d}t}(t) = \frac{\mathrm{d}\eta}{\mathrm{d}t}\left(\mu(t)\right)\cdot\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}t}\left(y^{-1}(t)\right)}\\ &= \frac{\mathrm{d}\eta}{\mathrm{d}t}\left(\mu(t)\right)\cdot\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}t}\left(\mu(t)\right)} = \frac{b^*(\eta(\mu(t)))}{m^*(\eta(\mu(t)))} = \frac{b^*(s(t))}{m^*(s(t))} = \frac{b^*(\gamma(t),v(\cdot,t))}{m^*(\gamma(t))},\end{aligned} \tag{2.2.4}$$

where the last equality (2.2.4) follows since we assume that the whole order book dynamics are driven by the best bid and ask price dynamics, in the sense defined by the random operator in Assumption 2.1.6. The solution of (2.2.4) is the limiting model $s(t) = s(\cdot,t) = (\gamma(t), v(\cdot,t))'$ i.e. the key quantities best bid [ask] prices $\gamma(t) = (b(t), a(t))'$ and the relative buy [sell] volume densities $v(\cdot,t) = (v_b(\cdot,t), v_s(\cdot,t))'$.

2.2.1 Convergence of the Scaled Prices

The state-dependence in our models is such that the inter arrival times are only dependent on the best bid and ask price and the best bid and ask price are in turn only dependent on themselves and each other. This enables us to solve for the limit prices and specify the inter arrival operator $m^*(\gamma)$ in (2.2.4). In other words, the convergence and specification of the prices may be shown separately and this is done in Proposition 2.2.3 below. For the proof of convergence and specification of the limiting volume densities, much additional work is needed and it is done in Section 2.2.2 below.

To show convergence of the scaled prices, we will need the following lemma that shows the convergence of the expected price dynamics.

Lemma 2.2.1 (Convergence of the expected price dynamics)**.** *Consider the following sequence*

$$\hat{\gamma}^{(n)}(t) := \hat{\gamma}^{(n)}_k = \begin{pmatrix} b^{(n)}_k \\ a^{(n)}_k \end{pmatrix} \quad \textit{for } t \in [t^{(n)}_k, t^{(n)}_{k+1}),$$

$$\begin{cases} \hat{\gamma}^{(n)}_{k+1} & := \hat{\gamma}^{(n)}_k + \mathbb{E}\left[\mathcal{D}^{(n)}_{\gamma,k}(\hat{\gamma}^{(n)}_k)\middle|\mathcal{F}^{\Phi}_{k-1}\right] \\ \hat{\gamma}^{(n)}_0 & = \begin{pmatrix} B_0 \\ A_0 \end{pmatrix}, \end{cases} \tag{2.2.5}$$

where the 2×1 *random operator* $\mathcal{D}^{(n)}_{\gamma,k}$ *is given by the price dynamics of* $\mathcal{D}^{(n)}_k$ *in (2.1.11) i.e.*

$$\mathcal{D}^{(n)}_{\gamma,k}(\hat{\gamma}^{(n)}_k) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \mathcal{D}^{(n)}_k(\hat{\gamma}^{(n)}_k) \tag{2.2.6}$$

and suppose that Assumption 2.1.6 holds. Then, for any $T > 0$

$$\sup_{t\in[0,T]} |\hat{\gamma}^{(n)}(t) - \hat{\gamma}(t)|_2 = \mathcal{O}(\Delta t^{(n)}),$$

where $\hat{\gamma}(t) = \begin{pmatrix} \hat{b}(t) \\ \hat{a}(t) \end{pmatrix}$ *with* $\hat{b}, \hat{a} \in C^2([0,T], \mathbb{R})$ *is the unique solution of the autonomous ODE*

$$\begin{cases} \frac{\mathrm{d}\hat{\gamma}(t)}{\mathrm{d}t} = A(\hat{\gamma}(t)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & t \in (0,T] \\ \\ \hat{\gamma}(0) = \begin{pmatrix} B_0 \\ A_0 \end{pmatrix} \end{cases} \tag{2.2.7}$$

and the matrix A *is given by (2.1.32) in Theorem 2.1.9.*

Proof. By (2.1.25) of Assumption 2.1.6 we have $p^{*,I}(\cdot,\cdot) \in C^{1,1}(\mathbb{R}\times\mathbb{R},(0,1))$ for $I = A, B, E, F$, where the derivatives are uniformly bounded. Thus, the $p^{*,I}$ are globally Lipschitz continuous functions and this implies global existence of a solution for (2.2.5), see Agarwal and O'Regan [3, Theorem 15.3 on p.107]. The process $\hat{\gamma}^{(n)}$ in (2.2.5) is deterministic and for the expectations of the price operators one has by (2.1.25) of Assumption 2.1.6

$$\begin{aligned} \mathbb{E}\left[\mathcal{D}^{(n)}_{\gamma,k}(\hat{\gamma}^{(n)}_k)\Big|\mathcal{F}^{\Phi}_{k-1}\right] &= \Delta x^{(n)} \begin{pmatrix} \mathbb{E}\left[\mathbb{1}^{(n),B}_k(\hat{\gamma}^{(n)}_k)\Big|\mathcal{F}^{\Phi}_{k-1}\right] - \mathbb{E}\left[\mathbb{1}^{(n),A}_k(\hat{\gamma}^{(n)}_k)\Big|\mathcal{F}^{\Phi}_{k-1}\right] \\ \mathbb{E}\left[\mathbb{1}^{(n),E}_k(\hat{\gamma}^{(n)}_k)\Big|\mathcal{F}^{\Phi}_{k-1}\right] - \mathbb{E}\left[\mathbb{1}^{(n),F}_k(\hat{\gamma}^{(n)}_k)\Big|\mathcal{F}^{\Phi}_{k-1}\right] \end{pmatrix} \\ &= \Delta t^{(n)} \cdot \frac{\Delta x}{\Delta t} \begin{pmatrix} p^{(n),A}(\hat{\gamma}^{(n)}_k) - p^{(n),B}(\hat{\gamma}^{(n)}_k) \\ p^{(n),E}(\hat{\gamma}^{(n)}_k) - p^{(n),F}(\hat{\gamma}^{(n)}_k) \end{pmatrix} \\ &= \Delta t^{(n)} \cdot \left\{ A(\hat{\gamma}^{(n)}_k) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + o(1) \right\}, \end{aligned} \tag{2.2.8}$$

where the last equality follows since $p^{(n),I} \to p^{*,I}$ uniformly as $n \to \infty$ by Assumption 2.1.6. One may regard the sequence (2.2.8) as a numerical scheme for the autonomous ODE (2.1.30). As such it is a special case of the classical Euler scheme and converges uniformly to the unique solution of (2.1.30) by the theorem in Hairer et. al [38, Theorem 7.3 on p.37 and p.51-54] with the rate of $\Delta t^{(n)}$. The C^2-property of the solutions follows by Olver [59, Theorem 20.10 on p.1104]. ☐

Lemma 2.2.2 (Global Lipschitz property of conditional expected price changes)**.** *Let*

$X, Y \in \mathbb{R}^2$ be $\mathcal{F}^{\Phi}_{k-1}$-measurable random variables and suppose that Assumption 2.1.6 holds.

Then, there exists a constant $L_\gamma > 0$ for all k and n, such that

$$\left| \mathbb{E}\left[\mathcal{D}^{(n)}_{\gamma,k}(X) \middle| \mathcal{F}^{\Phi}_{k-1} \right] - \mathbb{E}\left[\mathcal{D}^{(n)}_{\gamma,k}(Y) \middle| \mathcal{F}^{\Phi}_{k-1} \right] \right|_2 \le \Delta t^{(n)} \cdot L_\gamma \cdot |X - Y|_2 . \tag{2.2.9}$$

Proof. By (2.1.25) of Assumption 2.1.6, it holds that $\mathbb{E}\left[\mathbb{1}^{(n),I}_k(X) \middle| \mathcal{F}^{\Phi}_{k-1} \right] = \frac{p^{(n),I}(X^{(n)}_{\gamma,k})}{n^{s-1}}$ for all $I = A, B, E, F$ as X is $\mathcal{F}^{\Phi}_{k-1}$-measurable. Furthermore, $\Delta t^{(n)} = \frac{\Delta t}{n^s}$ and $\Delta x^{(n)} = \frac{\Delta x}{n}$ by our scaling assumptions and thus

$$\mathbb{E}\left[\mathcal{D}^{(n)}_{\gamma,k}(X) \middle| \mathcal{F}^{\Phi}_{k-1} \right] = \Delta x^{(n)} \begin{pmatrix} \mathbb{E}\left[\mathbb{1}^{(n),B}_k(\hat{\gamma}^{(n)}_k) \middle| \mathcal{F}^{\Phi}_{k-1} \right] - \mathbb{E}\left[\mathbb{1}^{(n),A}_k(\hat{\gamma}^{(n)}_k) \middle| \mathcal{F}^{\Phi}_{k-1} \right] \\ \mathbb{E}\left[\mathbb{1}^{(n),E}_k(\hat{\gamma}^{(n)}_k) \middle| \mathcal{F}^{\Phi}_{k-1} \right] - \mathbb{E}\left[\mathbb{1}^{(n),F}_k(\hat{\gamma}^{(n)}_k) \middle| \mathcal{F}^{\Phi}_{k-1} \right] \end{pmatrix}$$

$$= \Delta t^{(n)} \cdot \frac{\Delta x}{\Delta t} \begin{pmatrix} p^{(n),B}(X) - p^{(n),A}(X) \\ p^{(n),E}(X) - p^{(n),F}(X) \end{pmatrix} \tag{2.2.10}$$

This means that (2.2.9) holds since the expression in brackets on the right hand side of (2.2.10) is globally Lipschitz continuous in the $|\cdot|_2$-norm by the assumptions of the uniform bound of the first derivatives of the event probabilities $p^{(n),I}$ (see (2.1.25) in Assumption 2.1.6) and this ensures the global existence of the constant L_γ in (2.2.9). ☐

The following proposition corresponds to the convergence result for the prices contained in our main result Theorem 2.1.9.

Proposition 2.2.3 (Convergence of the best bid and ask prices)**.** *Suppose that Assumption 2.1.6 holds. Then, for all $T > 0$ we have that the best bid and ask prices*

$$\Gamma^{(n)}(t) := \begin{pmatrix} B^{(n)}(t) \\ A^{(n)}(t) \end{pmatrix}$$

in the sequence of order book models (2.1.11)-(2.1.10) converge

$$\sup_{t \in [0,T]} |\Gamma^{(n)}(t) - \gamma(t)|_2 \to 0, \quad a.s. \ as \ n \to \infty, \tag{2.2.11}$$

where $\gamma(t) = \begin{pmatrix} b(t) \\ a(t) \end{pmatrix}$ with $b, a \in C^2([0,T], \mathbb{R})$ denote the best bid and ask price in the limit model and solve the coupled ODE (2.1.30) of Theorem 2.1.9 uniquely.

Proof. We have assumed that the inter arrival times are a.s. positive and that $m^*(\gamma)$ is globally Lipschitz. Thus, $b, a \in C^2([0,T], \mathbb{R})$ solve (2.1.30) uniquely by the arguments in the beginning of the proof of Lemma 2.2.1.

By Proposition 4.2.6, we can consider the price process $\Gamma^{(n)}$ (random in state and time) as the composition of a state process $\eta_\gamma^{(n)}$ (random in the price state $\mathbb{R}^2$, deterministic jump times) and a time process $\mu^{(n)}$ (random in state=time $\mathbb{R}$, deterministic jump times) as follows

$$\Gamma^{(n)}(t) = \eta_\gamma^{(n)}\left(\mu^{(n)}(t) - \Delta t^{(n)}\right) \quad \text{for all } t, \tag{2.2.12}$$

with $\mu^{(n)}(t) := \inf\{u : u > 0, y^{(n)}(u) > t\}$ where

$$y^{(n)}(u) := \tau_k^{(n)} \quad \text{and} \quad \eta_\gamma^{(n)}(u) : = \Gamma_k^{(n)} \quad \text{for } u \in \left[t_k^{(n)}, t_{k+1}^{(n)}\right). \tag{2.2.13}$$

Since we wish to apply the time change Theorem 4.2.7 to show convergence of the composition, we study the limits of the processes $\eta_\gamma^{(n)}$ and $\mu^{(n)}$, separately.

We first consider the convergence of the state process $\eta_\gamma^{(n)}$ of the prices and claim that

$$\sup_{t\in[0,T]} |\eta_\gamma^{(n)}(t) - \hat{\gamma}(t)|_2 \to 0, \quad \text{a.s as } n \to \infty, \tag{2.2.14}$$

where $\hat{\gamma}$ solves (2.1.30) of Lemma 2.2.1.

From the price dynamics of (2.1.11) we have, by adding and subtracting the conditional expectation of the incremental price changes w.r.t. the event sub-σ-algebra $\mathcal{F}^\Phi$ that

$$\begin{aligned}\eta_\gamma^{(n)}(t) &= \gamma(0) + \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t}\rfloor} \mathcal{D}_{\gamma,k}^{(n)}\left(\eta_{\gamma,k}^{(n)}\right)\\ &= \gamma(0) + \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t}\rfloor} \mathbb{E}\left[\mathcal{D}_{\gamma,k}^{(n)}(\eta_{\gamma,k}^{(n)})\middle|\mathcal{F}_{k-1}^\Phi\right] + \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t}\rfloor}\left(\mathcal{D}_{\gamma,k}^{(n)}(\eta_{\gamma,k}^{(n)}) - \mathbb{E}\left[\mathcal{D}_{\gamma,k}^{(n)}(\eta_{\gamma,k}^{(n)})\middle|\mathcal{F}_{k-1}^\Phi\right]\right)\end{aligned} \tag{2.2.15}$$

Now, from (2.2.15), we get by adding and subtracting the sequence $\hat{\gamma}^{(n)}$ defined in Lemma 2.2.1:

$$\begin{aligned}\left|\eta_\gamma^{(n)}(t) - \hat{\gamma}(t)\right|_2 &= \left|\left(\eta_\gamma^{(n)}(t) - \hat{\gamma}^{(n)}(t)\right) + \left(\hat{\gamma}^{(n)}(t) - \hat{\gamma}(t)\right)\right|_2\\ &\leq \left|\sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t}\rfloor} \mathbb{E}\left[\mathcal{D}_{\gamma,k}^{(n)}(\eta_{\gamma,k}^{(n)})\middle|\mathcal{F}_{k-1}^\Phi\right] - \hat{\gamma}^{(n)}(t)\right|_2\end{aligned}$$

$$+\left|\sum_{k=0}^{\lfloor\frac{n^s t}{\Delta t}\rfloor}\left(\mathcal{D}_{\gamma,k}^{(n)}(\eta_{\gamma,k}^{(n)})-\mathbb{E}\left[\mathcal{D}_{\gamma,k}^{(n)}(\eta_{\gamma,k}^{(n)})\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right)\right|_2+\left|\hat{\gamma}^{(n)}(t)-\hat{\gamma}(t)\right|_2. \tag{2.2.16}$$

For the first term in (2.2.16), we have by (2.2.5) of Lemma 2.2.1 that

$$\begin{aligned}
&\left|\sum_{k=0}^{\lfloor\frac{n^s t}{\Delta t}\rfloor}\mathbb{E}\left[\mathcal{D}_{\gamma,k}^{(n)}(\eta_{\gamma,k}^{(n)})\middle|\mathcal{F}_{k-1}^{\Phi}\right]-\hat{\gamma}^{(n)}(t)\right|_2\\
&=\left|\sum_{k=0}^{\lfloor\frac{n^s t}{\Delta t}\rfloor}\mathbb{E}\left[\mathcal{D}_{\gamma,k}^{(n)}(\eta_{\gamma,k}^{(n)})\middle|\mathcal{F}_{k-1}^{\Phi}\right]-\mathbb{E}\left[\mathcal{D}_{\gamma,k}^{(n)}(\hat{\gamma}_k^{(n)})\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right|_2\\
&\leq\sum_{k=0}^{\lfloor\frac{n^s t}{\Delta t}\rfloor}\left|\mathbb{E}\left[\mathcal{D}_{\gamma,k}^{(n)}(\eta_{\gamma,k}^{(n)})\middle|\mathcal{F}_{k-1}^{\Phi}\right]-\mathbb{E}\left[\mathcal{D}_{\gamma,k}^{(n)}(\hat{\gamma}_k^{(n)})\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right|_2\\
&\leq\Delta t^{(n)}\cdot L_\gamma\cdot\sum_{k=0}^{\lfloor\frac{n^s t}{\Delta t}\rfloor}\left|\eta_{\gamma,k}^{(n)}-\hat{\gamma}_k^{(n)}\right|_2
\end{aligned}\tag{2.2.17}$$

which follows by Lemma 2.2.2 since $\eta_{\gamma,k}^{(n)}$ and $\hat{\gamma}_k^{(n)}$ are $\mathcal{F}_{k-1}^{\Phi}$-measurable.

The second term of (2.2.16) is the noise of the prices and we show almost sure convergence to zero as $n\to\infty$ by calculating the convergence rate using the Strong Law of Large Numbers Theorem 5.2.2 of the Appendix. For this matter, we denote

$$Y_k^{(n)}\left(\eta_{\gamma,k}^{(n)}\right):=\begin{pmatrix}Y_{b,k}^{(n)}\left(\eta_{\gamma,k}^{(n)}\right)\\ Y_{s,k}^{(n)}\left(\eta_{\gamma,k}^{(n)}\right)\end{pmatrix}=\mathcal{D}_{\gamma,k}^{(n)}(\eta_{\gamma,k}^{(n)})-\mathbb{E}\left[\mathcal{D}_{\gamma,k}^{(n)}(\eta_{\gamma,k}^{(n)})\middle|\mathcal{F}_{k-1}^{\Phi}\right],$$

where $Y_{b,k}^{(n)}$ [$Y_{s,k}^{(n)}$] denotes the bid [ask] price component. The sequence $\{Y_k^{(n)}\left(\eta_{\gamma,k}^{(n)}\right)\}_{k\geq 0}$ is a martingale difference array for all n, i.e. integrable, $\mathcal{F}_k$-measurable and

$$\begin{aligned}
\mathbb{E}\left[Y_{k+1}^{(n)}\left(\eta_{\gamma,k+1}^{(n)}\right)\middle|\mathcal{F}_k\right]&=\mathbb{E}\left[\mathcal{D}_{\gamma,k+1}^{(n)}(\eta_{\gamma,k+1}^{(n)})-\mathbb{E}\left[\mathcal{D}_{\gamma,k+1}^{(n)}(\eta_{\gamma,k+1}^{(n)})\middle|\mathcal{F}_k^{\Phi}\right]\middle|\mathcal{F}_k\right]\\
&=\mathbb{E}\left[\mathcal{D}_{\gamma,k+1}^{(n)}(\eta_{\gamma,k+1}^{(n)})\middle|\mathcal{F}_k\right]-\mathbb{E}\left[\mathcal{D}_{\gamma,k+1}^{(n)}(\eta_{\gamma,k+1}^{(n)})\middle|\mathcal{F}_k^{\Phi}\right]=0
\end{aligned}$$

as the random variables of $\mathcal{D}_{\gamma,k+1}^{(n)}(\eta_{\gamma,k+1}^{(n)})$ are conditionally independent and measurable w.r.t $\mathcal{F}_k^{\Phi}\subset\mathcal{F}_k$, respectively.

We will show the a.s. convergence for the bid price component (the ask price component convergence may be shown analogously). From the dynamics of the prices and

(2.1.25) of Assumption 2.1.6, we have

$$Y_{b,k}^{(n)}(\eta_{\gamma,k}^{(n)}) = \Delta x^{(n)} \left\{ \left(\mathbb{1}_k^{(n),B}(\eta_{\gamma,k}^{(n)}) - \mathbb{1}_k^{(n),A}(\eta_{\gamma,k}^{(n)}) \right) - \frac{1}{n^{s-1}} \left(p^{(n),B}(\eta_{\gamma,k}^{(n)}) - p^{(n),A}(\eta_{\gamma,k}^{(n)}) \right) \right\}. \tag{2.2.18}$$

We now claim that that there exist an a.s. finite random variable C_0 and a number $n_0 \in \mathbb{N}$ such that for $\delta \in (0,1)$

$$\left| \sum_{k=0}^{\lfloor \frac{n^s T}{\Delta t} \rfloor} Y_{b,k}^{(n)}(\eta_{\gamma,k}^{(n)}) \right| \leq C_0\, n^{-\frac{\delta}{2}} \quad \text{a.s. for all } n > n_0 \tag{2.2.19}$$

by applying Theorem 5.2.2 of the Appendix for terms with $k > 1$ ($Y_{b,0}^{(n)}(\eta_{\gamma,0}^{(n)})$ is easily seen to be $\mathcal{O}(n^{-1})$ a.s.). To better apply the result, we express (2.2.18), using $\Delta x^{(n)} = \frac{\Delta x}{n}$ as

$$Y_{b,k}^{(n)}(\eta_{\gamma,k}^{(n)}) = \frac{\Delta x}{n} X_{b,k}^{(n)}(\eta_{\gamma,k}^{(n)}),$$

where

$$X_{b,k}^{(n)}(\eta_{\gamma,k}^{(n)}) := \left(\mathbb{1}_k^{(n),B}(\eta_{\gamma,k}^{(n)}) - \mathbb{1}_k^{(n),A}(\eta_{\gamma,k}^{(n)}) \right) - \frac{1}{n^{s-1}} \left(p^{(n),B}(\eta_{\gamma,k}^{(n)}) - p^{(n),A}(\eta_{\gamma,k}^{(n)}) \right) \tag{2.2.20}$$

and

$$S_{\lfloor \frac{n^s T}{\Delta t} \rfloor}^{(n)} := \sum_{k=1}^{\lfloor \frac{n^s T}{\Delta t} \rfloor} X_k^{(n)}(\eta_{\gamma,k}^{(n)}). \tag{2.2.21}$$

We now use the form of (2.2.21) to show the claim (2.2.19). Since the summation index satisfies $k \leq \lfloor \frac{n^s T}{\Delta t} \rfloor \leq \frac{n^s T}{\Delta t}$ by assumption, it follows that $\frac{\Delta x}{n} \leq \frac{\Delta x \left(\frac{T}{\Delta t}\right)^{1/s}}{k^{1/s}}$ and

$$\left| \sum_{k=0}^{\lfloor \frac{n^s T}{\Delta t} \rfloor} Y_k^{(n)}(\eta_{\gamma,k}^{(n)}) \right| = \left| \frac{\Delta x}{n} S_{\lfloor \frac{n^s T}{\Delta t} \rfloor}^{(n)} \right| \leq \left| \frac{S_{\lfloor \frac{n^s T}{\Delta t} \rfloor}^{(n)}}{b_{\lfloor \frac{n^s T}{\Delta t} \rfloor}} \right|, \quad \text{where } b_k := \frac{k^{1/s}}{\Delta x \left(\frac{T}{\Delta t}\right)^{1/s}}. \tag{2.2.22}$$

Thus it is sufficient to consider the sequence $\frac{S_{\lfloor \frac{n^s T}{\Delta t} \rfloor}^{(n)}}{b_{\lfloor \frac{n^s T}{\Delta t} \rfloor}}$ and we will apply Theorem 5.2.2 of the Appendix for this sequence. $\left\{ X_k^{(n)}(\eta_{\gamma,k}^{(n)}) \right\}_{k \geq 0}$ is easily seen to be an $\mathcal{F}_k$-martingale difference sequence in $\mathbb{R}$ (see Definition 4.2.1). With the short hand notation $\mathbb{1}_k^{(n),I} = \mathbb{1}_k^{(n),I}\left(\eta_{\gamma,k}^{(n)}\right)$ and $p^{(n),I}\left(\eta_{\gamma,k}^{(n)}\right) = p^{(n),I}$ we have using the obvious uniform bounds of the event indicators and scaling of probabilities (see (2.1.25) of Assumption 2.1.6) that there exists an upper bound $\hat{C}_u < \infty$

$$
\begin{aligned}
&\mathbb{E}\left[\left|X_{b,k}^{(n)}(\eta_{\gamma,k}^{(n)})\right|^2\right] \\
&= \mathbb{E}\left[\left|\left(\mathbb{1}_k^{(n),B} - \mathbb{1}_k^{(n),A}\right) - \frac{1}{n^{s-1}}\left(p^{(n),B} - p^{(n),A}\right)\right|^2\right] \\
&= \mathbb{E}\left[\left(\mathbb{1}_k^{(n),B} - \mathbb{1}_k^{(n),A}\right)^2\right] - 2\mathbb{E}\left[\left(\mathbb{1}_k^{(n),B} - \mathbb{1}_k^{(n),A}\right)\right]\frac{1}{n^{s-1}}\left(p^{(n),B} - p^{(n),A}\right) \\
&\qquad\qquad + \left(\frac{1}{n^{s-1}}\left(p^{(n),B} - p^{(n),A}\right)\right)^2 \\
&= \mathbb{E}\left[\mathbb{1}_k^{(n),B} - 2\,\mathbb{1}_k^{(n),B}\mathbb{1}_k^{(n),A} + \mathbb{1}_k^{(n),A}\right] - \left(\frac{1}{n^{s-1}}\left(p^{(n),B} - p^{(n),A}\right)\right)^2 \\
&= \frac{p^{(n),B} + p^{(n),A}}{n^{s-1}} - \left(\frac{1}{n^{s-1}}\left(p^{(n),B} - p^{(n),A}\right)\right)^2 \\
&= \frac{1}{n^{s-1}}\left(p^{(n),B} + p^{(n),A} - \frac{1}{n^{s-1}}\left(\left(p^{(n),B} - p^{(n),A}\right)\right)^2\right) && (2.2.23) \\
&\le \frac{\hat{C}_u}{n^{s-1}} \le \underbrace{\hat{C}_u\left(\frac{T}{\Delta t}\right)^{1-1/s}}_{=:\hat{C}_{1,u}}\frac{1}{k^{1-1/s}}, \quad \text{for all } k \le \left\lfloor\frac{n^sT}{\Delta t}\right\rfloor && (2.2.24)
\end{aligned}
$$

Now, using b_k of (2.2.22), we set $\alpha_k := \frac{\hat{C}_{1,u}}{k^{1-1/s}}$ and have since $s > 1$ by assumption:

$$
\sum_{k=1}^{\infty}\frac{\alpha_k}{b_k^2} = \sum_{k=1}^{\infty}\left\{\frac{\hat{C}_{1,u}}{k^{1-1/s}}\cdot\frac{\Delta x^2\left(\frac{T}{\Delta t}\right)^{2/s}}{k^{2/s}}\right\} = \hat{C}_{1,u}\Delta x^2\left(\frac{T}{\Delta t}\right)^{2/s}\sum_{k=1}^{\infty}\frac{1}{k^{1+1/s}} < \infty. \quad (2.2.25)
$$

We want to prove the explicit convergence rate in (2.2.19). The corresponding quantities of (5.2.3) in Theorem 5.2.2 are, using Remark 5.2.3, given by

$$
\begin{aligned}
\nu_m &= \sum_{k=m}^{\infty}\frac{\alpha_k}{b_k^2} \le \hat{C}_{1,u}\Delta x^2\left(\frac{T}{\Delta t}\right)^{2/s}\left(\frac{1}{m^{1+1/s}} + \int_m^{\infty}\frac{1}{k^{1+1/s}}\mathrm{d}k\right) \\
&\qquad\qquad = \underbrace{\hat{C}_{1,u}\Delta x^2\left(\frac{T}{\Delta t}\right)^{2/s}s}_{=:\hat{C}_{2,u}}\cdot\left(\frac{\frac{1}{s}+m}{m^{1+1/s}}\right), \\
\beta_m &= \max_{1\le k\le m} b_k\left(\nu_k\right)^{\delta/2} \le \max_{1\le k\le m}\left\{\frac{k^{1/s}}{\Delta x\left(\frac{T}{\Delta t}\right)^{1/s}}\left(\hat{C}_{2,u}\left(\frac{\frac{1}{s}+k}{k^{1+1/s}}\right)\right)^{\delta/2}\right\} \\
&\qquad\qquad = \hat{C}_{3,u}\cdot\left(m^{\frac{1}{s}-\frac{\delta}{2}\left(1+\frac{1}{s}\right)}(m+\frac{1}{s})^{\delta/2}\right) =: \hat{\beta}_m, && (2.2.26)
\end{aligned}
$$

where $\hat{C}_{3,u} := \frac{\hat{C}_{2,u}^{\delta/2}}{\Delta x \left(\frac{T}{\Delta t}\right)^{1/s}}$. We have

$$\left| \frac{S^{(n)}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}}{b_{\lfloor \frac{n^s T}{\Delta t} \rfloor}} \right| \leq \sup_{n \geq 1} \left\{ \left| \frac{S^{(n)}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}}{\hat{\beta}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}} \right| \right\} \cdot \frac{\hat{\beta}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}}{b_{\lfloor \frac{n^s T}{\Delta t} \rfloor}},$$

where $\hat{\beta}_m$ is the upper bound defined in (2.2.26). Furthermore, assumption (5.2.1) of Theorem 5.2.2 holds: analogously as in (5.2.12) of Theorem 5.2.2 in the Appendix, there exists $C < \infty$ such that

$$\mathbb{E}\left[\left(\sup_{n \geq 1} \left| \frac{S^{(n)}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}}{\hat{\beta}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}} \right| \right)^2 \right] \leq 4 \cdot 4C \sum_{k=1}^{\infty} \frac{\alpha_k}{\hat{\beta}_k^2} = \frac{16 C \hat{C}_{1,u}}{\hat{C}_{3,u}^2} \sum_{k=1}^{\infty} \frac{1}{k^{(1+\frac{1}{s})(1-\delta)} \left(k + \frac{1}{s}\right)^{\delta}} < \infty,$$

$\delta \in (0,1)$ and thus the random variable $\hat{C}_0 := \sup_{n \geq 1} \left| \frac{S^{(n)}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}}{\hat{\beta}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}} \right| < \infty$ a.s. exists.

In summary, by (2.2.19) and the upper bound of (2.2.26), for $\delta \in (0,1)$ we have

$$\begin{aligned}
\left| \sum_{k=0}^{\lfloor \frac{n^s T}{\Delta t} \rfloor} Y^{(n)}_{b,k}(\eta^{(n)}_{\gamma,k}) \right| &\leq \left| Y^{(n)}_{b,0}(\eta^{(n)}_{\gamma,0}) \right| + \left| \frac{S^{(n)}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}}{b_{\lfloor \frac{n^s T}{\Delta t} \rfloor}} \right| \\
&\leq \left| Y^{(n)}_{b,0}(\eta^{(n)}_{\gamma,0}) \right| + \sup_{n \geq 1} \left\{ \left| \frac{S^{(n)}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}}{\hat{\beta}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}} \right| \right\} \frac{\hat{\beta}_{\lfloor \frac{n^s T}{\Delta t} \rfloor}}{b_{\lfloor \frac{n^s T}{\Delta t} \rfloor}} \\
&= \mathcal{O}(n^{-1}) + \hat{C}_0 \cdot \hat{C}_{2,u}^{\delta/2} \frac{\left(\lfloor \frac{n^s T}{\Delta t} \rfloor^{\frac{1}{s} - \frac{\delta}{2}\left(1 + \frac{1}{s}\right)} \left(\lfloor \frac{n^s T}{\Delta t} \rfloor + \frac{1}{s} \right)^{\delta/2} \right)}{\lfloor \frac{n^s T}{\Delta t} \rfloor^{1/s}} \\
&= \mathcal{O}\left(n^{-\frac{\delta}{2}}\right) \quad \text{a.s.}
\end{aligned} \tag{2.2.27}$$

and this implies (2.2.19). Thus, we have that

$$\begin{aligned}
&\left| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left(\mathcal{D}^{(n)}_{\gamma,k}(\eta^{(n)}_{\gamma,k}) - \mathbb{E}\left[\mathcal{D}^{(n)}_{\gamma,k}(\eta^{(n)}_{\gamma,k}) \middle| \mathcal{F}^{\Phi}_{k-1} \right] \right) \right|_2 \qquad (2.2.28) \\
&= \left| \Delta x^{(n)} \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\{ \begin{pmatrix} \mathbb{1}^{(n),B}_k(\eta^{(n)}_{\gamma,k}) - \mathbb{1}^{(n),A}_k(\eta^{(n)}_{\gamma,k}) \\ \mathbb{1}^{(n),E}_k(\eta^{(n)}_{\gamma,k}) - \mathbb{1}^{(n),F}_k(\eta^{(n)}_{\gamma,k}) \end{pmatrix} \right. \right. \\
&\qquad \left. \left. - \frac{1}{n^{s-1}} \begin{pmatrix} p^{(n),B}(\eta^{(n)}_{\gamma,k}) - p^{(n),A}(\eta^{(n)}_{\gamma,k}) \\ p^{(n),E}(\eta^{(n)}_{\gamma,k}) - p^{(n),F}(\eta^{(n)}_{\gamma,k}) \end{pmatrix} \right\} \right|_2
\end{aligned}$$

$$= \mathcal{O}\left(n^{-\frac{\delta}{2}}\right) = o(1) \quad \text{a.s.} \tag{2.2.29}$$

For the third term of (2.2.16), we have

$$\sup_{t\in[0,T]} |\hat{\gamma}^{(n)}(t) - \hat{\gamma}(t)|_2 = \mathcal{O}(\Delta t^{(n)}) \tag{2.2.30}$$

by Lemma 2.2.1. Hence, for (2.2.16) it holds that

$$\left|\eta_\gamma^{(n)}(t) - \hat{\gamma}(t)\right|_2 \leq \Delta t^{(n)} \cdot L_\gamma \cdot \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left|\eta_{\gamma,k}^{(n)} - \hat{\gamma}_k^{(n)}\right|_2 + o(1) \quad \text{a.s.}$$

One can now apply the discrete Gronwall Lemma 4.2.8 to show the uniform convergence of $|\eta_\gamma^{(n)}(t) - \hat{\gamma}(t)|_2 \to 0$ *a.s.* over $[0, T]$ and thus we have proved the convergence (2.2.14).

For the cumulative time process $y^{(n)}$ defined in (2.2.13), we have

$$\begin{aligned} y^{(n)}(t) &= \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} C_k^{(n)}\left(\eta_{\gamma,k}^{(n)}\right) \\ &= \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \mathbb{E}\left[C_k^{(n)}\left(\eta_{\gamma,k}^{(n)}\right)\middle|\mathcal{F}_{k-1}^{\Phi}\right] + \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left(C_k^{(n)}\left(\eta_{\gamma,k}^{(n)}\right) - \mathbb{E}\left[C_k^{(n)}\left(\eta_{\gamma,k}^{(n)}\right)\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right) \\ &= \Delta t^{(n)} \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} m^{(n)}\left(\eta_{\gamma,k}^{(n)}\right) + \Delta t^{(n)} \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left(\frac{C_k^{(n)}\left(\eta_{\gamma,k}^{(n)}\right)}{\Delta t^{(n)}} - m^{(n)}\left(\eta_{\gamma,k}^{(n)}\right)\right). \end{aligned} \tag{2.2.31}$$

Furthermore, by the above a.s. uniform convergence of $\eta_\gamma^{(n)} \to \hat{\gamma}$ and from Assumption 2.1.6 we have $m^{(n)}, m^* \in C^1$ and $m^{(n)} \to m^*$ uniformly as $n \to \infty$. Thus, the first sum in (2.2.31) converges to

$$y(t) = \Delta t \int_0^t m^*(\hat{\gamma}(u))\mathrm{d}u. \tag{2.2.32}$$

Applying the Strong Law of Large Numbers for martingale difference arrays (Theorem 5.2.2) to the second sum using the uniform condition (2.1.26) its a.s. convergence follows, we conclude that

$$|y^{(n)}(t) - y(t)| \to 0 \quad \text{a.s. as } n \to \infty.$$

$y^{(n)}$ and y are monotone increasing functions and thus the inverses $y^{(n),-1}$ and y^{-1} exist. From the definition of $\mu^{(n)}$ in (2.2.12)-(2.2.13), we have uniform convergence over the compact set $[0,T]$ since the limit is continuous, i.e.

$$\sup_{t\in[0,T]} |\mu^{(n)}(t)-\mu(t)| \to 0 \quad \text{a.s. as } n\to\infty$$

where $\mu(t)=y^{-1}(t)$ and thus we have

$$\mu'(t) = \left(y^{-1}(t)\right)' = \frac{1}{y'(y^{-1}(t))} = \frac{1}{m^*(\mu(t))} = \frac{1}{m^*\left(\hat{\gamma}(\mu(t))\right)}. \tag{2.2.33}$$

The best bid and ask prices in the models may be written as a composition $\Gamma^{(n)}(t) = \eta_\gamma^{(n)}\left(\eta^{(n)}(t)-\Delta t^{(n)}\right)$ as in (2.2.12) and applying the Time Change Theorem 4.2.7, since both the state and the time process converge, we conclude that

$$\sup_{t\in[0,T]} \left|\Gamma^{(n)}(t)-\gamma(t)\right|_2 \to 0 \quad \text{as } n\to\infty,$$

where $\gamma(t)=\hat{\gamma}(\mu(t))$ and by the chain rule, (2.2.7), (2.2.32) and (2.2.33) we have that

$$\gamma'(t) = \hat{\gamma}'(\mu(t))\cdot\mu'(t) = \frac{A(\hat{\gamma}(\mu(t)))}{m^*(\hat{\gamma}(\mu(t)))}\begin{pmatrix}1\\1\end{pmatrix} = \frac{A(\gamma(t))}{m^*(\gamma(t))}\begin{pmatrix}1\\1\end{pmatrix}$$

and thus γ solves (2.1.30) of Theorem 2.1.9. □

2.2.2 Convergence of the Scaled Relative Volume Densities

At this point, what remains to complete the proof of the main result is to show that the state process of the volume density functions

$$\eta_v^{(n)}(\cdot,t) := \eta_{v,k}^{(n)}, \quad \text{for } t\in[t_k^{(n)}, t_k^{(n+1)}), \tag{2.2.34}$$

where

$$\begin{cases} \eta_{v,k+1}^{(n)} & := \eta_k^{(n)} + \mathcal{D}_{v,k}^{(n)}\left(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)}\right) \\ \\ \eta_{v,0}^{(n)} & := u_0^{(n)} \end{cases} \tag{2.2.35}$$

and the 2×1 random operator $\mathcal{D}_{v,k}^{(n)}$ is given by the volume dynamics of $\mathcal{D}_k^{(n)}$ in (2.1.11) i.e.

$$\mathcal{D}_{v,k}^{(n)}\left(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)}\right) := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathcal{D}_k^{(n)}\left(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)}\right), \tag{2.2.36}$$

converges uniformly, i.e. that

$$\sup_{t\in[0,T]} \|\eta_v^{(n)}(\cdot,t) - u(\cdot,t)\|_v \to 0 \quad \text{a.s. as } n \to \infty, \tag{2.2.37}$$

where $u := \begin{pmatrix} u_b \\ u_s \end{pmatrix}$ with $u_b, u_s \in C^{2,2}(\mathbb{R} \times [0,T], \mathbb{R}_{>0})$ is the unique solution of the first-order linear hyperbolic PDE system

$$\begin{cases} u_t = A\left(\hat{\gamma}(t)\right) u_x + B\left(x, \hat{\gamma}(t)\right) u + c\left(x, \hat{\gamma}(t)\right), \\ \\ u(x,0) = v_0(x) \end{cases} \tag{2.2.38}$$

and the variable coefficients $A(\cdot)$, $B(x,\cdot)$ and $c(x,\cdot)$ are given by (2.1.32)-(2.1.34) in Theorem 2.1.9 and $\hat{\gamma}$ solves (2.2.7) in Lemma 2.2.1.

We will show the convergence by splitting up (2.2.37) in several terms and this will be done in detail below. An essential step will be to find a convergent discretization of the PDE system (2.2.38). There are of course many existing convergent numerical schemes for PDE:s of this kind but in our case the scheme must be coherent with the order book dynamics and the law of large numbers scaling. In Proposition 2.2.6 below, we propose a numerical scheme $u^{(n)}$, that is closely related to the expected order book dynamics for the density dynamics, and show that it converges uniformly to u. Our scaling of the price and time tick provides a mesh of the price-time interval for the volume densities.

Definition 2.2.4 (Mesh of the price-time interval)**.** *The price and time ticks of the n:th model determine the mesh of the price-time space $(-\infty,\infty) \times [0,T]$. For this purpose, let $\Delta x^{(n)}$ be the mesh length in the x-variable and $\Delta t^{(n)}$ be the mesh length in the time variable. We set*

$$x_j^{(n)} := j\Delta x^{(n)}, \quad \textit{for } j \in \mathbb{Z} \quad \textit{and}$$

$$t_k^{(n)} := k\Delta t^{(n)}, \quad \textit{for } k = 0, \ldots, \left\lfloor \frac{n^s T}{\Delta t} \right\rfloor.$$

For fixed n, our scaling defines a grid function approximation $u_k^{(n),j}$ to $u(x_j^{(n)}, t_k^{(n)})$ over the mesh in Definition 2.2.4. We have for all j,k,n that $u_k^{(n),j}$ is a step function of step length $\Delta x^{(n)}$ (i.e. one price tick) and its height is the average volume density over the step length. The numerical scheme $u^{(n)}$ is a so called finite volume method due to the scaling of the initial densities (see (2.1.4)) and we use the standard arguments of consistency and stability (see Leveque [54, Chapter 10] and Leveque [55, p.140-146]) to show its uniform convergence in the $L^2(\mathbb{R},\mathbb{R})$-norm.

Remark 2.2.5. *The grid in the plots of Figures 1.4-1.7 may be interpreted as that of a price and time tick grid. In other words, as the ticks get smaller, the mesh gets finer and a continuous model becomes a viable approximation of the LOB.*

Proposition 2.2.6 (Convergence of the numerical scheme)**.** *Let the mesh of the price-time interval be that of Definition 2.2.4, suppose that Assumptions 2.1.6 and 2.1.8 hold and consider the deterministic process in $t \in [0,T]$ such that*

$$u^{(n)}(\cdot,t) := u_k^{(n)}(\cdot), \quad \textit{for } t \in [t_k^{(n)}, t_{k+1}^{(n)}) \tag{2.2.39}$$

with

$$u_k^{(n)} := \begin{pmatrix} u_{b,k}^{(n)} \\ u_{s,k}^{(n)} \end{pmatrix}, \quad u_k^{(n),j} := u_k^{(n)}(x_j^{(n)}) \quad \textit{and} \quad \begin{cases} u_{k+1}^{(n),j} & := \mathcal{H}^{(n)}\left(u_k^{(n),j}\right) \\ \\ u_0^{(n),j} & := v_0^{(n)}(x_j^{(n)}), \end{cases} \tag{2.2.40}$$

with $v_0^{(n)} := \begin{pmatrix} v_{b,0}^{(n)} \\ v_{s,0}^{(n)} \end{pmatrix}$ as in the initial densities (2.1.4) and the operator $\mathcal{H}^{(n)}$ is such that

$$\begin{aligned} \mathcal{H}^{(n)}(u_k^{(n),j};x) := u_k^{(n),j} + \frac{A_M(t_k^{(n)})}{n^{s-1}}(u_k^{(n),j+1} - u_k^{(n),j}) + \frac{A_S(t_k^{(n)})}{n^{s-1}}(u_k^{(n),j-1} - u_k^{(n),j}) \\ + \frac{1}{n^{s-1}}F(x,t_k^{(n)})u_k^{(n),j} + \frac{1}{n^{s-1}}g(x,t_k^{(n)}), \end{aligned} \tag{2.2.41}$$

where we have that the coefficients are given by

$$A_M(t_k^{(n)}) := \begin{pmatrix} p^{*,A}(\hat{\gamma}(t_k^{(n)})) & 0 \\ 0 & p^{*,E}(\hat{\gamma}(t_k^{(n)})) \end{pmatrix}, \tag{2.2.42}$$

$$A_S(t_k^{(n)}) := \begin{pmatrix} p^{*,B}(\hat{\gamma}(t_k^{(n)})) & 0 \\ 0 & p^{*,F}(\hat{\gamma}(t_k^{(n)})) \end{pmatrix}, \quad \textit{where} \tag{2.2.43}$$

$$A_M(t_k^{(n)}) - A_S(t_k^{(n)}) = \frac{\Delta t}{\Delta x}A(\hat{\gamma}(t_k^{(n)})), \tag{2.2.44}$$

$$F(x,t_k^{(n)}) := \Delta t B(x,\hat{\gamma}(t_k^{(n)})), \quad g(x,t_k^{(n)}) := \Delta t c(x,\hat{\gamma}(t_k^{(n)})) \tag{2.2.45}$$

and the matrices $A(\cdot)$, $B(x,\cdot)$ and the vector $c(x,\cdot)$ are given by (2.1.32)-(2.1.34) in Theorem 2.1.9.

Then, the process $u^{(n)}$ defined by (2.2.39)-(2.2.45) is a convergent finite difference scheme of the PDE:s (2.2.38), i.e. we have that

$$\sup_{t\in[0,T]} \|u^{(n)}(\cdot,t) - u(\cdot,t)\|_v = \mathcal{O}\left(\Delta x^{(n)}\right), \tag{2.2.46}$$

where $\|(u_b,u_s)'\|_v = \|u_b\|_{L^2} + \|u_s\|_{L^2}$ and $u := \begin{pmatrix} u_b \\ u_s \end{pmatrix}$ with $u_b, u_s \in C^{2,2}(\mathbb{R}\times[0,T],\mathbb{R}_{>0})$ is the unique solution of (2.2.38) over $[0,T]$.

Proof. We first establish that the PDE system (2.2.38) has unique $C^{2,2}$-solutions. The equations for the buy and sell side can be solved independently since they are coupled only through the best bid and ask prices. For the buy-side PDE, we can write

$$\begin{cases} \frac{\partial u_b}{\partial t} = A_b(t)\frac{\partial u_b}{\partial x} + B_b(x,t)u_b + c_b(x,t) \\ u_b(x,0) = v_{b,0}(x) \end{cases} \tag{2.2.47}$$

Using the method of characteristic curves, the PDE reduces to a family of ODE:s; see Evans [29, Chapter 3] for details or Polyanin et al. [65] for the general solution formula. The characteristic equations for our buy-side PDE read

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}\tau} = -A_b(\tau) \\ x(0) = \xi \end{cases} \quad \text{and} \quad \begin{cases} \frac{\mathrm{d}\bar{u}_b}{\mathrm{d}\tau} = B_b(x(\tau),\tau)\bar{u}_b + c_b(x(\tau),\tau) \\ \bar{u}_b(\xi,0) = v_{b,0}(\xi). \end{cases} \tag{2.2.48}$$

The solution of this system of linear ODE:s as a function of the state $\xi \in \mathbb{R}$ can be given in closed form:

$$\begin{aligned} x(t,\xi) &= \xi - \int_0^t A_b(t)\mathrm{d}t \\ \bar{u}_b(\xi,t) &= e^{\int_0^t B_b(x(u,\xi),u)\mathrm{d}u}\left\{ v_{b,0}(\xi) + \int_0^t e^{-\int_0^s B_b(x(u,\xi),u)\mathrm{d}u} \cdot c_b(x(s,\xi),s)\mathrm{d}s \right\}. \end{aligned}$$

It describes the surface $\{(t,\xi) : u_b(x(t,\xi),t) = \bar{u}_b(\xi,t) \text{ given } u_b(\xi,0) = v_{b,0}(\xi)\}$. The solution to the buy-side PDE can be recovered from the unique solution to the ODE-system through

$$u_b(x,t) = \bar{u}_b\Big(x + \int_0^t A_b(s)ds, t\Big). \tag{2.2.49}$$

Due to our smoothness assumptions on the model parameters it is not hard to verify that the first and second derivatives of the solution exist.

We proceed and show consistency in the $L^2(\mathbb{R},\mathbb{R})$-norm. The local truncation error for the numerical method using the operator $\mathcal{H}^{(n)}$ is defined as

$$\mathcal{L}_b^{(n)}(x,t) := \frac{1}{\Delta t^{(n)}}\left(u_b(x,t+\Delta t^{(n)}) - \mathcal{H}^{(n)}(u_b(\cdot,t);x)\right). \tag{2.2.50}$$

For the local truncation error $\mathcal{L}_b^{(n)}$ (2.2.50), we have via substituting the buy side part of the operator $\mathcal{H}^{(n)}$ (2.2.41) (denoted $A_{b,M},\ldots,g_b$), smoothness of the solution (2.2.49) and Taylor expansion with Lagrange remainders (denoted $R_{x+,t}(\Delta x^{(n)})$, $R_{x-,t}(\Delta x^{(n)})$ and $R_{x,t+}(\Delta t^{(n)})$), using the equality (2.2.44) along with the scaling assumptions $\Delta x^{(n)} = \frac{\Delta x}{n}$ as well as $\Delta t^{(n)} = \frac{\Delta t}{n^s}$ and finally that u_b solves (2.2.47):

$$\begin{aligned}
\mathcal{L}_b^{(n)}(x,t) = \frac{1}{\Delta t^{(n)}}\Big(u_b(x,t+\Delta t^{(n)}) - u_b(x,t) &- \frac{A_{b,M}(t)}{n^{s-1}}(u_b(x+\Delta x^{(n)},t) - u_b(x,t))\\
&- \frac{A_{b,S}(t)}{n^{s-1}}(u_b(x-\Delta x^{(n)},t) - u_b(x,t))\\
&- \frac{1}{n^{s-1}}F_b(x,t)u_b(x,t) - \frac{1}{n^{s-1}}g_b(x,t)\Big)
\end{aligned}$$

$$\begin{aligned}
= \frac{1}{\Delta t^{(n)}}\Big(\frac{\partial u_b}{\partial t}(x,t)\Delta t^{(n)} + R_{x,t+}(\Delta t^{(n)}) &- \frac{A_{b,M}(t)}{n^{s-1}}(\frac{\partial u_b}{\partial x}(x,t)\Delta x^{(n)} + R_{x+,t}(\Delta x^{(n)}))\\
&- \frac{A_{b,S}(t)}{n^{s-1}}(-\frac{\partial u_b}{\partial x}(x,t)\Delta x^{(n)} - R_{x-,t}(\Delta x^{(n)})\\
&- \frac{1}{n^{s-1}}F_b(x,t)u(x,t) - \frac{1}{n^{s-1}}g(x,t)\Big)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial u_b}{\partial t}(x,t) - \frac{\Delta x}{\Delta t}\left(A_M(t) - A_S(t)\right)\frac{\partial u_b}{\partial x}(x,t) - \frac{1}{\Delta t}F_b(x,t)u_b(x,t) - \frac{1}{\Delta t}g_b(x,t)\\
&\qquad + \frac{R_{x,t+}(\Delta t^{(n)})}{\Delta t^{(n)}} - \frac{\Delta x A_{b,M}(t)}{\Delta t}\frac{R_{x+,t}(\Delta x^{(n)})}{\Delta x^{(n)}} + \frac{\Delta x A_{b,S}(t)}{\Delta t}\frac{R_{x-,t}(\Delta x^{(n)})}{\Delta x^{(n)}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\partial u_b}{\partial t}(x,t) - A_b(t)\frac{\partial u_b}{\partial x}(x,t) - B_b(x,t)u_b(x,t) - c_b(x,t)\\
&+ \frac{\partial^2 u_b}{\partial t^2}(x,t+c_1\Delta t^{(n)})\frac{\left(\Delta t^{(n)}\right)^2}{2\Delta t^{(n)}}\\
&+ \left(-\frac{\Delta x A_{b,M}(t)}{\Delta t}\frac{\partial^2 u_b}{\partial x^2}(x+c_2\Delta x^{(n)},t) + \frac{\Delta x A_{b,S}(t)}{\Delta t}\frac{\partial^2 u_b}{\partial x^2}(x-c_3\Delta x^{(n)},t)\right)\frac{(\Delta x^{(n)})^2}{2\Delta x^{(n)}}
\end{aligned}$$

$$\begin{aligned}
\overset{(2.2.47)}{=} \frac{1}{2}\Bigg[\frac{\partial^2 u_b}{\partial t^2}\left(x,t+c_1\Delta t^{(n)}\right)\Delta t^{(n)} + \Bigg(&-\frac{\Delta x A_{b,M}(t)}{\Delta t}\frac{\partial^2 u_b}{\partial x^2}(x+c_2\Delta x^{(n)},t)\\
&+ \frac{\Delta x A_{b,S}(t)}{\Delta t}\frac{\partial^2 u_b}{\partial x^2}(x-c_3\Delta x^{(n)},t)\Bigg)\Delta x^{(n)}\Bigg]
\end{aligned} \tag{2.2.51}$$

where $c_i \in (0,1)$.

Studying the $C^{2,2}$-solution (2.2.49), and recalling that placement/cancelation only occurs over a finite interval $[-M,M]$ in the space variable we reason as follows. Due to smooth price shifts over the compact time interval $[0,T]$, the initial density $v_{b,0}$ will only be effected by placement/cancelation within a compact space interval $[-M_T, M_T]$. Outside of this interval the solution is just a shifted version of the initial volume density function. This reasoning is useful for estimating the integrability of the second derivatives in the space variable, appearing in the local truncation error (2.2.51). Considering the interval $[-M_T, M_T]$ and its compliment we have

$$\begin{aligned}
&\|\mathcal{L}_b^{(n)}(x,t)\|_{L^2(\mathbb{R},\mathbb{R})} \\
&= \left(\int_{-\infty}^{\infty} |\mathcal{L}_b^{(n)}(x,t)|^2 \mathrm{d}x\right)^{1/2} \\
&= \left(\int_{-\infty}^{-M_T} |\mathcal{L}_b^{(n)}(x,t)|^2 \mathrm{d}x + \int_{-M_T}^{M_T} |\mathcal{L}_b^{(n)}(x,t)|^2 \mathrm{d}x + \int_{M_T}^{\infty} |\mathcal{L}_b^{(n)}(x,t)|^2 \mathrm{d}x\right)^{1/2} \\
&= \left(\mathcal{O}\Big((\Delta x^{(n)})^2\Big) \left[K_{v_{b,0}-} + 2M_T K_{u_{b,xx},u_{b,tt}} + K_{v_{b,0}+}\right]\right)^{1/2} = \mathcal{O}\left(\Delta x^{(n)}\right) \qquad (2.2.52)
\end{aligned}$$

where the constants in the tails exist since $v''_{b,0} \in L^2(\mathbb{R},\mathbb{R})$ by assumption 2.1.5 and $A_b(t)$ as well as its derivative are assumed to be bounded by Assumption 2.1.6. As the second derivatives are continuous the integral over the compact interval $[-M_T, M_T]$ may easily be bounded by the mean value theorem for integrals. By analogous arguments the same estimate holds for the local truncation error of the sell side $\mathcal{L}_s^{(n)}(x,t)$ and thus the numerical scheme is L^2-consistent with the accuracy of the price tick $\Delta x^{(n)}$.

We define the vector-valued error-function $E^{(n)}(x,t) := u^{(n)}(x,t) - u(x,t)$ and we have for its value at the next time grid point using the operator $\mathcal{H}^{(n)}$:

$$E^{(n)}(x, t + \Delta t^{(n)}) = \mathcal{H}^{(n)}(u^{(n)}(\cdot,t);x) - u(x, t+\Delta t^{(n)}). \qquad (2.2.53)$$

From (2.2.50), we have

$$u(x, t+\Delta t^{(n)}) = \mathcal{H}^{(n)}(u(\cdot,t);x) + \Delta t^{(n)} \mathcal{L}^{(n)}(x,t). \qquad (2.2.54)$$

Using (2.2.53) and (2.2.54) it follows that

$$\begin{aligned}
E^{(n)}(x, t+\Delta t^{(n)}) &= \mathcal{H}^{(n)}(u^{(n)}(\cdot,t);x) - \mathcal{H}^{(n)}(u(\cdot,t);x) - \Delta t^{(n)} \mathcal{L}^{(n)}(x,t) \\
&= \mathcal{H}^{(n)}(E^{(n)}(\cdot,t);x) - \Delta t^{(n)} \mathcal{L}^{(n)}(x,t),
\end{aligned}$$

where the last equality follows by the linearity of the operator $\mathcal{H}^{(n)}$. Using this property iteratively, at time $t_k^{(n)}$ we have for the error function

$$E^{(n)}(\cdot, t_k^{(n)}) = (\mathcal{H}^{(n)})^k (E^{(n)})(\cdot, 0) - \Delta t^{(n)} \sum_{i=1}^{k} (\mathcal{H}^{(n)})^{k-i} (\mathcal{L}^{(n)}(\cdot, t_{i-1})). \qquad (2.2.55)$$

Now, we show convergence via stability in the sense of Lax-Richtmyer for our method, via the sufficient condition that

$$\|\mathcal{H}^{(n)}\| = \sup_{\|U\|_v=1} \|\mathcal{H}^{(n)}(U)\|_v \leq 1 + K\Delta t^{(n)}, \quad \text{for some } K \in [0,\infty)$$

where $\|\mathcal{H}^{(n)}(u_k^{(n)})\|_v = \|u_{k+1}^{(n)}\|_v = \|u_{b,k+1}^{(n)}\|_{L^2} + \|u_{s,k+1}^{(n)}\|_{L^2}$. From (2.2.41) we have, with the short hand notation $p^{*,I} := p^{*,I}(\hat{\gamma}(t_k^{(n)}))$, $I \in \{A,\dots,D\}$, collecting terms, the fact that $\frac{p^{*,A}}{n^{s-1}} + \frac{p^{*,B}}{n^{s-1}} + \frac{\mu_{\omega C}}{n^{s-1}} p^{*,C} f^C \leq 1$ (since the $p^{*,I}$ sum to unity and by the assumption of proportional partial cancelation), using the triangle inequality, the isometric property of translation operators $T_{\pm}^{(n)}$ (Part iii) of Lemma 5.2.4) and collecting terms again:

$$\begin{aligned}
&\|u_{b,k+1}^{(n)}\|_{L^2} \\
&= \Big\| u_{b,k}^{(n)} + \left(T_+^{(n)}\left(u_{b,k}^{(n)}\right) - u_{b,k}^{(n)}\right)\frac{p^{*,A}}{n^{s-1}} + \left(T_-^{(n)}\left(u_{b,k}^{(n)}\right) - u_{b,k}^{(n)}\right)\frac{p^{*,B}}{n^{s-1}} \\
&\quad - \frac{\mu_{\omega C}}{n^s} p^{*,C} f^C u_{b,k}^{(n)} + \frac{\mu_{\omega D}}{n^s} p^{*,D} f^D \Big\|_{L^2} \\
&= \Big\| \underbrace{\left(1 - \left(\frac{p^{*,A}}{n^{s-1}} + \frac{p^{*,B}}{n^{s-1}} + \frac{\mu_{\omega C}}{n^s} p^{*,C} f^C\right)\right)}_{\geq 0} u_{b,k}^{(n)} + T_+^{(n)}\left(u_{b,k}^{(n)}\right)\frac{p^{*,A}}{n^{s-1}} \\
&\quad + T_-^{(n)}\left(u_{b,k}^{(n)}\right)\frac{p^{*,B}}{n^{s-1}} + \frac{\mu_{\omega D}}{n^s} p^{*,D} f^D \Big\|_{L^2} \\
&\leq \left(1 - \left(\frac{p^{*,A}}{n^{s-1}} + \frac{p^{*,B}}{n^{s-1}} + \frac{\mu_{\omega C}}{n^s} p^{*,C}\right)\right) \left\|u_{b,k}^{(n)}\right\|_{L^2} + \frac{p^{*,A}}{n^{s-1}} \left\|T_+^{(n)}\left(u_{b,k}^{(n)}\right)\right\|_{L^2} \\
&\quad + \frac{p^{*,B}}{n^{s-1}} \left\|T_-^{(n)}\left(u_{b,k}^{(n)}\right)\right\|_{L^2} + \left\|\frac{\mu_{\omega D}}{n^s} p^{*,D} f^D\right\|_{L^2} \\
&= \underbrace{\left(1 - \frac{\mu_{\omega C}}{n^{s-1}} p^{*,C}\right)}_{\leq 1} \left\|u_{b,k}^{(n)}\right\|_{L^2} + \Delta t^{(n)} \frac{1}{\Delta t} \mu_{\omega D} p^{*,D} K_{f^D}
\end{aligned}$$

where $K_{f^I} := \|f^I\|_{L^2}$ exists by the C^2-property and compactness assumption of (2.1.28). We get the analogous result for the sell side, hence

$$\begin{aligned}
\|\mathcal{H}^{(n)}(u_k^{(n)})\|_v &\leq \|u_{b,k}^{(n)}\|_{L^2}^2 + \|u_{s,k}^{(n)}\|_{L^2}^2 + \underbrace{\frac{1}{\Delta t}\Big(\mu_{\omega^D} p^{*,D} K_{f^D} + \mu_{\omega^H} p^{*,H} K_{f^H}\Big)}_{=:K\geq 0} \Delta t^{(n)} \\
&= \|u_k^{(n)}\|_v + K\Delta t^{(n)}
\end{aligned}$$

and one has $\|\mathcal{H}^{(n)}\| = \sup_{\|U\|_v=1} \|\mathcal{H}^{(n)}(U)\|_v \leq 1 + K\Delta t^{(n)}$ where the constant K is bounded since $p^{*,A}, \ldots, p^{*,H} \in (0,1)$, $\mu_\omega^D, \mu_\omega^H \in (0,\infty)$ and f^D, f^H are L^2-integrable.

Now, taking the norm of the error function (2.2.55) and utilizing part ii) of Lemma 5.2.5 in the Appendix:

$$\begin{aligned}\|E^{(n)}(\cdot,0)\|_v = \|u_0^{(n)}(\cdot) - v_0(\cdot)\|_v &= \|v_{b,0}^{(n)}(\cdot) - v_{b,0}(\cdot)\|_{L^2} + \|v_{s,0}^{(n)}(\cdot) - v_{s,0}(\cdot)\|_{L^2} \\ &= \mathcal{O}(\Delta x^{(n)}),\end{aligned}$$

$(1+K\Delta t^{(n)})^{k-i} \leq (1+K\Delta t^{(n)})^{\frac{n^s t}{\Delta t}} \leq e^{Kt}$ and that the local error is $o(1)$ by (2.2.51), we indeed have convergence:

$$\begin{aligned}&\|u^{(n)}(\cdot,t) - u(\cdot,t)\|_v \\ &= \|E^{(n)}(\cdot,t_k^{(n)})\|_v \\ &\leq \|(\mathcal{H}^{(n)})^{\lfloor\frac{n^s t}{\Delta t}\rfloor}\| \cdot \|E^{(n)}(\cdot,0)\|_v + \Delta t^{(n)} \sum_{i=1}^{\lfloor\frac{n^s t}{\Delta t}\rfloor} \|(\mathcal{H}^{(n)})^{k-i}\| \cdot \|\mathcal{L}^{(n)}(\cdot,t_{i-1}^{(n)})\|_v \\ &\leq e^{Kt}\left(\mathcal{O}(\Delta t^{(n)}) + \Delta t^{(n)}\mathcal{O}(\lfloor\frac{n^s t}{\Delta t}\rfloor)\mathcal{O}(\Delta x^{(n)})\right) = \mathcal{O}(\Delta x^{(n)})\end{aligned}$$

which holds for all $t \in [0,T]$ and the estimate for $t = T$ is the upper bound so we are done. □

To show the convergence of the density we proceed stepwise and compare the random states of the density volume function $\eta_v^{(n)}$ with the deterministic numerical scheme $u^{(n)}$ in Proposition 2.2.6 by introducing the deterministic step function valued processes $\widetilde{u}^{(n)}$ and $\hat{u}^{(n)}$ such that for $t \in [0,T]$

$$\widetilde{u}^{(n)}(\cdot,t) := \widetilde{u}_k^{(n)}, \quad \text{for } t \in [t_k^{(n)}, t_k^{(n+1)}), \tag{2.2.56}$$

and

$$\hat{u}^{(n)}(\cdot,t) := \hat{u}_k^{(n)}, \quad \text{for } t \in [t_k^{(n)}, t_k^{(n+1)}), \tag{2.2.57}$$

where

$$\begin{cases} \widetilde{u}_{k+1}^{(n),j} & := \widetilde{u}_k^{(n),j} + \mathbb{E}\left[\mathcal{D}_{v,k}^{(n)}\left(\eta_{\gamma,k}^{(n)}, \widetilde{u}_k^{(n),j}\right)\Big|\mathcal{F}_{k-1}^{\Phi}\right] \\ \widetilde{u}_0^{(n),j} & := v_0^{(n)}(x_j^{(n)}) \end{cases} \tag{2.2.58}$$

and

$$\begin{cases} \hat{u}_{k+1}^{(n),j} &:= \hat{u}_k^{(n),j} + \mathbb{E}\left[\mathcal{D}_{v,k}^{(n)}\left(\hat{\gamma}(t_k^{(n)}), \hat{u}_k^{(n),j}\right) \Big| \mathcal{F}_{k-1}^{\Phi}\right] \\ \\ \hat{u}_0^{(n),j} &:= v_0^{(n)}(x_j^{(n)}), \end{cases} \tag{2.2.59}$$

respectively, with $\mathcal{D}_{v,k}^{(n)}$ being the volume dynamics defined in (2.2.36).

We continue and split up the quantity that is to converge in (2.2.46) by adding and subtracting the processes $\widetilde{u}^{(n)}$ and $\hat{u}^{(n)}$ defined in (2.2.56)-(2.2.59) above:

$$\begin{aligned} \|\eta_v^{(n)}(\cdot,t) - u(\cdot,t)\|_v \leq \|\eta_v^{(n)}(\cdot,t) - \widetilde{u}^{(n)}(\cdot,t)\|_v + \|\widetilde{u}^{(n)}(\cdot,t) - \hat{u}^{(n)}(\cdot,t)\|_v \\ + \|\hat{u}^{(n)}(\cdot,t) - u^{(n)}(\cdot,t)\|_v + \|u^{(n)}(\cdot,t) - u(\cdot,t)\|_v. \end{aligned} \tag{2.2.60}$$

The last term in (2.2.60) converges uniformly over compact time intervals to 0 by Proposition 2.2.6. It remains to show the convergence of the first three terms and this will be done in Propositions 2.2.10-2.2.12 below.

We have that

$$\|\eta_v^{(n)}(\cdot,t)\|_v = \|\left(\eta_{v_b}^{(n)}(\cdot,t), \eta_{v_s}^{(n)}(\cdot,t)\right)'\|_v = \|\eta_{v_b}^{(n)}(\cdot,t)\|_{L^2} + \|\eta_{v_s}^{(n)}(\cdot,t)\|_{L^2}. \tag{2.2.61}$$

and often it is sufficient to show the various convergences for one side, say the buy side since the densities are not dependent on the state of the other. The proof of the sell side convergence is then analogous.

The following lemma shows the impact of two different time scales. The first part states that over the time interval $[0,T]$ the number of price changes will a.s. be of order $\mathcal{O}(n)$. The second part of the lemma provides the rate of convergence in the L^2-norm of the volume placement fluctuations.

Lemma 2.2.7. *Let $t \in (0,T]$ and suppose that Assumptions 2.1.6 and 2.1.8 hold. We have that*

i) Given $\lfloor \frac{n^s t}{\Delta t} \rfloor$ events, the order of the number of price changing events is a.s. $\mathcal{O}(n)$, i.e. for $I = A, B, E, F$, all $\epsilon > 0$ and $K \in (0,\infty)$ let

$$A_n := \left\{ \omega \in \Omega : \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \mathbb{1}_k^{(n),I}\left(\eta_{\gamma,k}^{(n)}\right)(\omega) \geq K n^{1+\epsilon} \right\} \quad \textit{then} \quad \mathbb{P}(\limsup_{n\to\infty} A_n) = 0. \tag{2.2.62}$$

ii) The following growth rate holds a.s. for volume placements, events $I = D, H$:

$$\left\| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\{ M_{v,k}^{(n),I} \mathbb{1}_k^{(n),I} \left(\eta_{\gamma,k}^{(n)} \right) - \mathbb{E}\left[M_{v,k}^{(n),I} \mathbb{1}_k^{(n),I} \left(\eta_{\gamma,k}^{(n)} \right) \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\} \right\|_{L^2} = \mathcal{O}\left(n^{-\frac{\delta}{2}(s-1)} \right) \tag{2.2.63}$$

Proof. To prove i) we make a Borel-Cantelli argument (see Theorem 5.1.4 in the Appendix) and show that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. Let $\kappa > 0$ be fixed and let r be a number such that $r > \frac{1+\kappa}{\epsilon}$, then by the Markov inequality, one has

$$\begin{aligned} \mathbb{P}(A_n) = \mathbb{P}\left(\sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \mathbb{1}_k^{(n),I} \left(\eta_{\gamma,k}^{(n)} \right) \geq K n^{1+\epsilon} \right) &\leq \frac{\mathbb{E}\left[\left(\sum_{k=0}^{\lfloor \frac{n^s T}{\Delta t} \rfloor} \mathbb{1}_k^{(n),I} \left(\eta_{\gamma,k}^{(n)} \right) \right)^r \right]}{(K n^{1+\epsilon})^r} \\ &\leq \frac{\frac{n^s T}{\Delta t} \cdot \frac{1}{n^{s-1}} + \left(\left(\frac{n^s T}{\Delta t} \right)^r - \frac{n^s T}{\Delta t} \right) \cdot \left(\frac{1}{n^{s-1}} \right)^r}{K^r n^{r+r\epsilon}} = \mathcal{O}\left(\frac{1}{n^{r\epsilon}} \right) = \mathcal{O}\left(\frac{1}{n^{1+\kappa}} \right) \end{aligned} \tag{2.2.64}$$

and thus $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ which proves i). In (2.2.64), we have used that for the square of the sum there are $\frac{n^s T}{\Delta t}$ non-mixed terms and the rest are mixed terms of indicator functions. The expectations of the terms follow readily by the independence properties and condition (2.1.25) of Assumption 2.1.6.

We show ii) by arguing over the added height of the volume densities (step functions) in the book and using the L^2-norm for these step functions[6]. For this matter, denote the added height for $[x_j^{(n)}, x_{j+1}^{(n)})$ with

$$Y_k^{(n),j}(\eta_{\gamma,k}^{(n)}) := \frac{\omega_k^I}{n^s} \frac{\mathbb{1}_{\{\pi_k^D \in [x_j^{(n)}, x_{j+1}^{(n)})\}}}{\Delta x^{(n)}} \mathbb{1}_k^{(n),I} \left(\eta_{\gamma,k}^{(n)} \right) - \frac{\mu_{\omega_k^I}}{n^s} f^{(n),I}(x_j^{(n)}) p^{(n),I} \left(\eta_{\gamma,k}^{(n)} \right) \tag{2.2.65}$$

We now claim that that there exist a.s. finite random variables $C_{0,j}$ and a number $n_0 \in \mathbb{N}$ such that for $\delta \in (0,1)$

$$\left| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} Y_k^{(n),j}(\eta_{\gamma,k}^{(n)}) \right| \leq C_{0,j} \cdot n^{-\frac{\delta}{2}(s-1)} \quad \text{a.s. for all } n > n_0 \text{ and each } j \in \mathbb{Z} \tag{2.2.66}$$

by utilizing Theorem 5.2.2 of the Appendix for terms with $k > 1$ ($Y_0^{(n),j}(\eta_{\gamma,0}^{(n)})$ is easily seen to be $\mathcal{O}(n^{-(s-1)})$ a.s. for all j). To better apply the theorem, we express (2.2.65) as

$$Y_k^{(n),j}(\eta_{\gamma,k}^{(n)}) = \frac{1}{n^{s-2}} X_k^{(n),j}(\eta_{\gamma,k}^{(n)}) \tag{2.2.67}$$

[6] For the initial buy volume density one has e.g. $\|v_{b,0}^{(n)}\|_{L^2} = \left(\Delta x^{(n)} \sum_{j=-\infty}^{\infty} \left| v_{b,0}^{(n),j} \right|^2 \right)^{\frac{1}{2}}$.

$$X_k^{(n),j}(\eta_{\gamma,k}^{(n)}) := \frac{1}{n}\left(\omega_k^I \frac{\mathbb{1}_{\{\pi_k^I \in [x_j^{(n)}, x_{j+1}^{(n)})\}}}{\Delta x}\mathbb{1}_k^{(n),I}\left(\eta_{\gamma,k}^{(n)}\right) - \mu_{\omega_k^I}\frac{\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} f^I(x)\mathrm{d}x}{\Delta x} p^{(n),I}\left(\eta_{\gamma,k}^{(n)}\right)\right) \tag{2.2.68}$$

and define

$$S_{\lfloor \frac{n^s t}{\Delta t}\rfloor}^{(n),j} := \sum_{k=1}^{\lfloor \frac{n^s t}{\Delta t}\rfloor} X_k^{(n),j}(\eta_{\gamma,k}^{(n)}). \tag{2.2.69}$$

We now use (2.2.69) to show the claim (2.2.66). Since the summation index satisfies $k \leq \left\lfloor \frac{n^s T}{\Delta t}\right\rfloor \leq \frac{n^s T}{\Delta t}$, it follows that $\frac{1}{n^{s-2}} \leq \frac{\left(\frac{T}{\Delta t}\right)^{(s-2)/s}}{k^{(s-2)/s}} = \frac{\left(\frac{T}{\Delta t}\right)^{1-(2/s)}}{k^{1-(2/s)}}$ and

$$\left|\sum_{k=1}^{\lfloor \frac{n^s t}{\Delta t}\rfloor} Y_k^{(n),j}(\eta_{\gamma,k}^{(n)})\right| = \left|\frac{1}{n^{s-2}} S_{\lfloor \frac{n^s t}{\Delta t}\rfloor}^{(n),j}\right| \leq \left|\frac{S_{\lfloor \frac{n^s t}{\Delta t}\rfloor}^{(n),j}}{b_{\lfloor \frac{n^s t}{\Delta t}\rfloor}}\right|, \quad \text{where } b_k := \frac{k^{1-(2/s)}}{\left(\frac{T}{\Delta t}\right)^{1-(2/s)}}. \tag{2.2.70}$$

Thus it is sufficient to consider $\frac{S_{\lfloor \frac{n^s t}{\Delta t}\rfloor}^{(n),j}}{b_{\lfloor \frac{n^s t}{\Delta t}\rfloor}}$ and we will apply Theorem 5.2.2 of the Appendix for this sequence. $\left\{X_k^{(n),j}(\eta_{\gamma,k}^{(n)})\right\}_{k\geq 0}$ is easily seen to be an $\mathcal{F}_k$-martingale difference sequence in $\mathbb{R}$ for each n (see Definition 4.2.1). With the short hand notation $\mathbb{1}_k^{(n),I} = \mathbb{1}_k^{(n),I}\left(\eta_{\gamma,k}^{(n)}\right)$ and $p^{(n),I}\left(\eta_{\gamma,k}^{(n)}\right) = p^{(n),I}$ we have by the definition of $X_k^{(n),j}$ in (2.2.68) and the fact that the interval $[x_j^{(n)}, x_{j+1}^{(n)})$ has Lebesgue measure $\Delta x^{(n)} = \frac{\Delta x}{n}$ that

$$\begin{aligned}
&\mathbb{E}\left[\left|X_k^{(n),j}(\eta_{\gamma,k}^{(n)})\right|^2\right] \\
&= \frac{1}{n^2}\mathbb{E}\left[\left|\omega_k^I \frac{\mathbb{1}_{\{\pi_k^I \in [x_j^{(n)}, x_{j+1}^{(n)})\}}}{\Delta x}\mathbb{1}_k^{(n),I} - \mu_{\omega_k^I}\frac{\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} f^I(x)\mathrm{d}x}{\Delta x} p^{(n),I}\right|^2\right] \\
&= \frac{1}{n^2}\left(\mathbb{E}\left[\left(\omega_k^I\right)^2\right]\frac{\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} f^I(x)\mathrm{d}x}{(\Delta x)^2}\mathbb{E}\left[\mathbb{1}_k^{(n),I}\right] + \mu_{\omega_k^I}^2\left(\frac{\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} f^I(x)\mathrm{d}x}{\Delta x}p^{(n),I}\right)^2\right. \\
&\qquad\left. - 2\left(\mu_{\omega_k^I}\right)^2\left(\frac{\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} f^I(x)\mathrm{d}x}{\Delta x}\right)^2 \mathbb{E}\left[\mathbb{1}_k^{(n),I}\right] p^{(n),I}\right)
\end{aligned} \tag{2.2.71}$$

$$\leq \frac{1}{n^3}\mathbb{E}\left[\left(\omega_k^I\right)^2\right]\frac{f_{\max}}{\Delta x} \leq \underbrace{\mathbb{E}\left[\left(\omega_k^I\right)^2\right]\frac{f_{\max}}{\Delta x}\left(\frac{T}{\Delta t}\right)^{3/s}}_{=:\hat{C}_{1,u}<\infty}\frac{1}{k^{3/s}} = \frac{\hat{C}_{1,u}}{k^{3/s}}. \tag{2.2.72}$$

for $k \leq \left\lfloor \frac{n^s T}{\Delta t}\right\rfloor$ and all j. The uniform constant $\hat{C}_{1,u}$ exists as the second moments of the volumes ω_k^I exist by assumption, the integrals over the densities $f_{\min} \leq f^I \leq f_{\max}$ (by Assumption 2.1.8 the f^I are continuous and non-zero on a compact interval) are over equidistant interval lengths $\frac{\Delta x}{n}$, the indicators and probabilities are uniformly bounded. Thus, using b_m as in (2.2.70) and (2.2.72) we set $\alpha_k := \frac{\hat{C}_{1,u}}{k^{3/s}}$ and get

$$\sum_{k=1}^{\infty}\frac{\alpha_k}{b_k^2} = \hat{C}_{1,u}\left(\frac{T}{\Delta t}\right)^{2-(4/s)}\sum_{k=1}^{\infty}\frac{1}{k^{2-(1/s)}} < \infty, \quad \text{for all } j. \tag{2.2.73}$$

Calculating the quantities of (5.2.3) in Theorem 5.2.2 one has using Remark 5.2.3, that:

$$\begin{aligned}
\nu_m &= \sum_{k=m}^{\infty}\frac{\alpha_k}{b_k^2} \leq \hat{C}_{1,u}\left(\frac{T}{\Delta t}\right)^{2-(4/s)}\left(\frac{1}{m^{2-(1/s)}} + \int_m^{\infty}\frac{1}{k^{2-(1/s)}}\mathrm{d}k\right)\\
&= \underbrace{\hat{C}_{1,u}\left(\frac{T}{\Delta t}\right)^{2-(4/s)}\frac{s}{s-1}}_{=:\hat{C}_{2,u}<\infty}\left(\frac{\frac{s-1}{s}}{m^{2-(1/s)}} + \frac{1}{m^{1-(1/s)}}\right) = \hat{C}_{2,u}\cdot\left(\frac{\frac{s-1}{s}+m}{m^{2-(1/s)}}\right),
\end{aligned}$$

$$\begin{aligned}
\beta_m &= \max_{1\leq k\leq m} b_k\left(\nu_k\right)^{\delta/2} \leq \max_{1\leq k\leq m}\frac{k^{1-(2/s)}}{\left(\frac{T}{\Delta t}\right)^{1-(2/s)}}\left(\frac{\hat{C}_{2,u}\left(\frac{s-1}{s}+k\right)}{k^{2-(1/s)}}\right)^{\delta/2}\\
&= \hat{C}_{3,u}m^{\left(1-(2/s)-\delta[1-(1/2s)]\right)}\left(\frac{s-1}{s}+m\right)^{\delta/2} := \hat{\beta}_m,
\end{aligned} \tag{2.2.74}$$

where $\hat{C}_{3,u} := \frac{\hat{C}_{2,u}^{\delta/2}}{\left(\frac{T}{\Delta t}\right)^{1-(2/s)}}$.

We have

$$\left|\frac{S^{(n),j}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}{b_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}\right| \leq \sup_{n\geq 1}\left\{\left|\frac{S^{(n),j}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}{\hat{\beta}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}\right|\right\}\cdot\frac{\hat{\beta}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}{b_{\lfloor\frac{n^s t}{\Delta t}\rfloor}},$$

where $\hat{\beta}_m$ is the upper bound defined in (2.2.74). Analogously as in (5.2.12) of Theorem

5.2.2, there exists $C < \infty$ s.t.

$$\mathbb{E}\left[\left(\sup_{n\geq 1}\left|\frac{S^{(n),j}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}{\hat{\beta}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}\right|\right)^2\right] \leq 4\cdot 4C\sum_{k=1}^{\infty}\frac{\alpha_k}{\hat{\beta}_k^2}$$

$$= \frac{16C\hat{C}_{1,u}}{\hat{C}_{3,u}^2}\sum_{k=1}^{\infty}\frac{1}{k^{2-(1/s)-2\delta[1-(1/2s)]}\left(\frac{s-1}{s}+k\right)^{\delta}} < \infty \qquad (2.2.75)$$

for $\delta \in (0,1)$ and thus a random variable $\hat{C}_{0,j} := \sup_{n\geq 1}\left|\frac{S^{(n),j}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}{\hat{\beta}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}\right| < \infty$ a.s. exists for all j. We we will sum over the index j below, when we calculate the L^2-norm.

Consequently, by (2.2.70) we have

$$\left|\sum_{k=0}^{\lfloor\frac{n^s t}{\Delta t}\rfloor} Y_k^{(n),j}(\eta^{(n)}_{\gamma,k})\right|$$

$$\leq \left|Y_0^{(n),j}(\eta^{(n)}_{\gamma,0})\right| + \left|\frac{S^{(n),j}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}{b_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}\right|$$

$$\leq \left|Y_0^{(n),j}(\eta^{(n)}_{\gamma,0})\right| + \sup_{n\geq 1}\left\{\left|\frac{S^{(n),j}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}{\hat{\beta}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}\right|\right\}\frac{\hat{\beta}_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}{b_{\lfloor\frac{n^s t}{\Delta t}\rfloor}}$$

$$\leq \mathcal{O}\left(n^{-(s-1)}\right) + \hat{C}_{0,j}\cdot\frac{\left(\frac{T}{\Delta t}\right)^{1-(2/s)}\hat{C}_{3,u}\lfloor\frac{n^s t}{\Delta t}\rfloor^{\left(1-(2/s)-\delta[1-(1/2s)]\right)}\left(\frac{s-1}{s}+\lfloor\frac{n^s t}{\Delta t}\rfloor\right)^{\delta/2}}{\lfloor\frac{n^s t}{\Delta t}\rfloor^{1-(2/s)}}$$

$$\leq \mathcal{O}\left(n^{-(s-1)}\right)$$

$$+\hat{C}_{0,j}\cdot\hat{C}_{3,u}\left(\frac{T}{\Delta t}\right)^{1-(2/s)}\lfloor\frac{n^s t}{\Delta t}\rfloor^{-\delta\left(1-(1/2s)\right)}\sup_{n\geq 1}\left\{\left(\frac{s-1}{\lfloor\frac{n^s t}{\Delta t}\rfloor s}+1\right)^{\delta/2}\right\}\lfloor\frac{n^s t}{\Delta t}\rfloor^{-\delta/2}$$

$$= \mathcal{O}\left(n^{-(s-1)}\right) + \hat{C}_{0,j}\cdot\hat{C}_{3,u}\left(\frac{T}{\Delta t}\right)^{1-(2/s)}\left(2-\frac{1}{s}\right)^{\delta/2}\lfloor\frac{n^s t}{\Delta t}\rfloor^{-\frac{\delta}{2}\left(1-(1/s)\right)}$$

$$\leq C_{0,j}\cdot n^{-\frac{\delta}{2}\left(s-1\right)} \quad \text{a.s.} \qquad (2.2.76)$$

for all j which implies (2.2.66) with

$$C_{0,j} := \hat{C}_{0,j}\hat{C}_{3,u}\left(\frac{T}{\Delta t}\right)^{1-(2/s)}\left(2-\frac{1}{s}\right)^{\delta/2} + \left|Y_0^{(n),j}(\eta^{(n)}_{\gamma,0})\right|,$$

where the random variable $\left|Y_0^{(n),j}(\eta_{\gamma,0}^{(n)})\right| < \infty$ a.s. by Assumption 2.1.6.

Thus, by integrating over all intervals $\left\{[x_j^{(n)}, x_{j+1}^{(n)})\right\}_{j\in\mathbb{Z}}$, the step function property of the volume densities and since orders are a.s. placed within the distance $[-M, M]$ by Assumption 2.1.8, we have

$$\begin{aligned}
&\left\| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\{ M_{v,k}^{(n),I} \mathbb{1}_k^{(n),I}\left(\eta_{\gamma,k}^{(n)}\right) - \mathbb{E}\left[M_{v,k}^{(n),I} \mathbb{1}_k^{(n),I}\left(\eta_{\gamma,k}^{(n)}\right) \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\} \right\|_{L^2} \\
&= \left(\int_{-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} \left\{ \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} Y_k^{(n),j}(\eta_{\gamma,k}^{(n)}) \right\} \mathbb{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(x) \right)^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&= \left(\Delta x^{(n)} \sum_{j=-\frac{M}{\Delta x^{(n)}}}^{\frac{M}{\Delta x^{(n)}}} \left\{ \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} Y_k^{(n),j}(\eta_{\gamma,k}^{(n)}) \right\}^2 \right)^{\frac{1}{2}} \\
&\leq \left(\Delta x^{(n)} \sum_{j=-\frac{M}{\Delta x^{(n)}}}^{\frac{M}{\Delta x^{(n)}}} \left\{ C_{0,j} \cdot n^{-\frac{\delta}{2}(s-1)} \right\}^2 \right)^{\frac{1}{2}} \qquad \text{a.s. by (2.2.76) for all } j \\
&= n^{-\frac{\delta}{2}(s-1)} \underbrace{\left(\frac{\Delta x}{n} \sum_{j=-\frac{nM}{\Delta x}}^{\frac{nM}{\Delta x}} C_{0,j}^2 \right)^{\frac{1}{2}}}_{=:\hat{C} < \infty \text{ a.s.}} = \hat{C}^{1/2} \cdot n^{-\frac{\delta}{2}(s-1)} \quad \text{a.s.}
\end{aligned} \tag{2.2.77}$$

The existence of the random variable $\hat{C} < \infty$ a.s. is given since

$$\mathbb{E}[\hat{C}] = \frac{\Delta x}{n} \sum_{j=-\frac{nM}{\Delta x}}^{\frac{nM}{\Delta x}} \mathbb{E}\left[C_{j,0}^2\right] \leq \frac{\Delta x}{n} \frac{nM}{\Delta x} \widetilde{K} = 2M\widetilde{K} < \infty,$$

where the uniform upper bound $\widetilde{K}$ for the $\mathbb{E}\left[C_{j,0}^2\right]$ exists by (2.2.75) and and the claim follows. ☐

Remark 2.2.8. *The convergence rate in Lemma 2.2.7 ii) is given by*

$$\mathcal{O}\left(n^{-\frac{\delta}{2}(s-1)}\right).$$

We will need it to be $\mathcal{O}\left(n^{-1}\right)$, given our scaling rate of $\mathcal{O}(n)$ for the active orders, to be able to ue the SLLN on Banach spaces by Hoffman-Jorgensen and Pisier (Theorem 4.2.4). This yields the simple condition

$$-\frac{\delta}{2}(s-1) = -1 \quad \Leftrightarrow s = 1 + \frac{2}{\delta}.$$

In this sense our choice of scaling rate for the passive orders is optimal.

Before we show the convergence of the remaining three terms of (2.2.60), we state and prove a lemma which shows useful bounds for the volume densities and the conditional expectation of the volume operator. The lemma will come in handy later. From the upper and lower bound (2.2.78) of its first part, it follows that the volume densities remain in L^2 and that the limit order book is never empty.

Lemma 2.2.9 (Bounds for volumes and expected density changes)**.** *Consider the state process of the volume density functions*

$$\eta_{v,k}^{(n)} = \begin{pmatrix} \eta_{v_b,k}^{(n)} \\ \eta_{v_s,k}^{(n)} \end{pmatrix}$$

defined in (2.2.35). For the buy side (and analogously for the sell side) one has that for all $n > 1$ after $k \leq \lfloor \frac{n^s t}{\Delta t} \rfloor$ events, where $t \in [0, T]$:

i) There exists a lower and an upper bound bound for the relative volume densities:

$$\underline{K}_{\eta_v} \leq \|\eta_{v,k}^{(n)}\|_v \leq \overline{K}_{\eta_v} \quad \text{a.s. where } \underline{K}_{\eta_v}, \overline{K}_{\eta_v} \in (0, \infty). \tag{2.2.78}$$

ii) The norm of the conditional expectation of an incremental shift difference due to a price change is given by

$$\left\| \mathbb{E}\left[T_{\pm}^{(n)}\left(\eta_{v,k}^{(n)}\right) - \eta_{v,k}^{(n)} \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\|_v \leq \overline{K}_{T,\eta_v} \Delta x^{(n)} \quad \text{where } \overline{K}_{T,\eta_v} \in (0, \infty).$$

iii) If $X \in \mathbb{R}^2$ is $\mathcal{F}_{k-1}^{\Phi}$-measurable, there exists a constant $L_v > 0$ such that

$$\left\| \mathbb{E}\left[\mathcal{D}_{v,k}^{(n)}\left(X, \eta_{v,k}^{(n)}\right) \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\|_{L^2} \leq \Delta t^{(n)} \cdot L_v \cdot \left\| \eta_{v,k}^{(n)} \right\|_{L^2} \tag{2.2.79}$$

a.s. for all k and n.

Proof. For notational simplicity we suppress the argument of the event indicator functions in this proof i.e. $\mathbb{1}_k^{(n),I} = \mathbb{1}_k^{(n),I}(\eta_{\gamma,k}^{(n)})$ for $I = A, \ldots, H$ in the following.

We use the translation operators, consider the buy and sell side separately and claim that the state of the relative buy volume density after k events has the representation

$$\eta_{v_b,k}^{(n)}$$
$$= \left(\left(T_+^{(n)}\right)^{\sum_{i=0}^{k-1} \mathbb{1}_i^{(n),A}} \circ \left(T_-^{(n)}\right)^{\sum_{i=0}^{k-1} \mathbb{1}_i^{(n),B}} \right) \left(v_{b,0}^{(n)}\right)$$

$$
+\sum_{i=0}^{k-1}\left\{\left(\left(T_+^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbb{1}_j^{(n),A}}\circ\left(T_-^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbb{1}_j^{(n),B}}\right)\left(-M_{v,i}^{(n),C}\eta_{v_b,i}^{(n)}\mathbb{1}_i^{(n),C}+M_{v,i}^{(n),D}\mathbb{1}_i^{(n),D}\right)\right\}. \tag{2.2.80}
$$

The relation (2.2.80) holds by induction over k. For $k=0$, no events have occured. We have that the state is equal to its initial value and the relation (2.2.80) reads $\eta_{v_b,0}^{(n)}=v_{b,0}^{(n)}$. If one assumes that (2.2.80) holds for all $k=p$, then it also holds for $k=p+1$ since the recurrent definition of the dynamics (2.1.11), a simple re-write and applying the induction hypothesis along with the fact that the translation operators are linear (see Lemma 5.2.4) yields

$$
\begin{aligned}
&\eta_{v_b,p+1}^{(n)}\\
&=\eta_{v_b,p}^{(n)}+\left(T_+^{(n)}(\eta_{v_b,p}^{(n)})-\eta_{v_b,p}^{(n)}\right)\mathbb{1}_p^{(n),A}+\left(T_-^{(n)}(\eta_{v_b,p}^{(n)})-\eta_{v_b,p}^{(n)}\right)\mathbb{1}_p^{(n),B}\\
&\qquad-M_{v,p}^{(n),C}\eta_{v_b,p}^{(n)}\mathbb{1}_p^{(n),C}+M_{v,p}^{(n),D}\mathbb{1}_p^{(n),D}\\
&=\left(\left(T_+^{(n)}\right)^{\mathbb{1}_p^{(n),A}}\circ\left(T_-^{(n)}\right)^{\mathbb{1}_p^{(n),B}}\right)\left(\eta_{v_b,p}^{(n)}\right)-M_p^{(n),C}\eta_{v_b,p}^{(n)}\mathbb{1}_p^{(n),C}+M_p^{(n),D}\mathbb{1}_p^{(n),D}\\
&=\left(\left(T_+^{(n)}\right)^{\sum_{i=0}^{p}\mathbb{1}_i^{(n),A}}\circ\left(T_-^{(n)}\right)^{\sum_{i=0}^{p}\mathbb{1}_i^{(n),B}}\right)\left(v_{b,0}^{(n)}\right)\\
&+\sum_{i=0}^{p}\left\{\left(\left(T_+^{(n)}\right)^{\sum_{j=i+1}^{p}\mathbb{1}_j^{(n),A}}\circ\left(T_-^{(n)}\right)^{\sum_{j=i+1}^{p}\mathbb{1}_j^{(n),B}}\right)\left(-M_{v,i}^{(n),C}\eta_{v_b,i}^{(n)}\mathbb{1}_i^{(n),C}+M_{v,i}^{(n),D}\mathbb{1}_i^{(n),D}\right)\right\},
\end{aligned}
$$

which is analogous to the representation in (2.2.80).

We show the upper bound of i) using relation (2.2.80). The isometric property of the translation operators implies that a (generous) upper bound is achieved if we assume only volume placements. We get by adding and subtracting the conditional expectation of the added volume density and applying the triangle inequality, that

$$
\begin{aligned}
&\left\|\eta_{v_b}^{(n)}(\cdot,t)\right\|_{L^2}\\
&\leq\left\|v_{b,0}^{(n)}\right\|_{L^2}+\left\|\sum_{k=0}^{\lfloor\frac{n^s t}{\Delta t}\rfloor}\mathbb{E}\left[\underbrace{M_{v,k}^{(n),D}\mathbb{1}_k^{(n),D}}_{=:Z_k^{(n)}}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right\|_{L^2}+\left\|\sum_{k=0}^{\lfloor\frac{n^s t}{\Delta t}\rfloor}Z_k^{(n)}-\mathbb{E}\left[Z_k^{(n)}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right\|_{L^2}\\
&\leq\left\|v_{b,0}^{(n)}\right\|_{L^2}+n^s\frac{T}{\Delta t}\frac{1}{n^s}K_{M^D}+\mathcal{O}\left(n^{-\frac{\delta}{2}\left(s-1\right)}\right)\quad\text{a.s.} &(2.2.81)\\
&=\left\|v_{b,0}^{(n)}\right\|_{L^2}+\frac{T}{\Delta t}\left(K_{M^D}+\mathcal{O}\left(\frac{1}{n}\right)\right)\leq\overline{K}_{\eta_{v_b}}\quad\text{a.s.}
\end{aligned}
$$

where $\overline{K}_{\eta_{v_b}}$ exists since $\left\|v_{i,0}^{(n)}\right\|_{L^2} \leq \overline{K}_{v_i,0}$ for $i = b, s$ by part i) of Lemma 5.2.5 in the Appendix. In (2.2.81) we get the finite constant K_{M^D} (ω_k^D is a.s. bounded and $f^{(n),D}$ is integrable) and we get the third term by applying part ii) of Lemma 2.2.7 with $m = \frac{n^s T}{\Delta t}$ in (2.2.63). For the sell side we have an analogous bound $\overline{K}_{\eta_{v_s}}$ and we may choose $\overline{K}_{\eta_v} := \overline{K}_{\eta_{v_b}} + \overline{K}_{\eta_{v_s}}$.

To show that a lower bound exists, we have that $\left\|\eta_{v_b,k}^{(n)}\right\|_{L^2}$ gets smaller only if there is cancelation i.e. in case of event C when considering the buy side. Recalling the definition of the relative volume cancelation dynamics (2.1.15) we have that the change in norm due to limit order volume cancelation is given by

$$\|M_{v,k}^{(n),C}\eta_{v_b,k}^{(n)}\|_{L^2} = \left\|\frac{1}{n^s}\omega_k^C \sum_{j=-\infty}^{\infty} \frac{\mathbb{1}_{\{x,\pi_k^I \in [x_j^{(n)}, x_{j+1}^{(n)})\}}}{\Delta x^{(n)}} \eta_{v_b,k}^{(n)}\right\|_{L^2} < \frac{1}{n^s}\|\eta_{v_b,k}^{(n)}\|_{L^2} \quad \text{a.s.} \quad (2.2.82)$$

since $\omega_k^C < 1$ a.s. by assumption and the bound is strict as it is given by assuming cancelation at every tick simultaneously. By the reverse triangle inequality and (2.2.82), we have

$$\begin{aligned}
\left\|\eta_{v_b,k}^{(n)}\right\|_{L^2} \geq \left\|\eta_{v_b,k-1}^{(n)} - M_{v,k-1}^{(n),C}\eta_{v_b,k-1}^{(n)}\right\|_{L^2} &\geq \left|\|\eta_{v_b,k-1}^{(n)}\|_{L^2} - \|M_{v,k-1}^{(n),C}\eta_{v_b,k-1}^{(n)}\|_{L^2}\right| \\
&\geq \|\eta_{v_b,0}^{(n)}\|_{L^2}\left(1 - \frac{1}{n^s}\right)^k \quad \text{a.s.} \quad k \leq \frac{n^s T}{\Delta t} \\
&\geq \|v_{b,0}^{(n)}\|_{L^2}\left(1 - \frac{1}{n^s}\right)^{n^s \frac{T}{\Delta t}} \\
&\geq \left(\frac{15}{16}\right)^{16\frac{T}{\Delta t}} \|v_{b,0}^{(n)}\|_{L^2} \quad \text{a.s.} \qquad (2.2.83)
\end{aligned}$$

since $\lim_{n\to\infty}\left(1 - \frac{1}{n^s}\right)^{n^s} \nearrow e^{-1}$ the lower bound $\left(\frac{15}{16}\right)^{16}$ is achieved for $n = 2$. Hence it follows, as $\left\|v_{i,0}^{(n)}\right\|_{L^2} \geq \underline{K}_{v_i,0}$ for $i = b, s$ by part i) of Lemma 5.2.5 in the Appendix, that a lower bound for (2.2.78) is given by $\underline{K}_{\eta_v} := \left(\frac{15}{16}\right)^{16\frac{T}{\Delta t}} (\underline{K}_{v_b,0} + \underline{K}_{v_s,0})$.

We now show ii). By linearity of the translation operator $T_+^{(n)}$ and relation (2.2.80) we have that

$$\begin{aligned}
&T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} \\
&= \left(\left(T_+^{(n)}\right)^{\sum_{i=0}^{k-1}\mathbb{1}_i^{(n),A}} \circ \left(T_-^{(n)}\right)^{\sum_{i=0}^{k-1}\mathbb{1}_i^{(n),B}}\right)\left(T_+^{(n)}\left(v_{b,0}^{(n)}\right) - v_{b,0}^{(n)}\right) \\
&- \sum_{i=0}^{k-1}\left(\left(T_+^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbb{1}_j^{(n),A}} \circ \left(T_-^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbb{1}_j^{(n),B}}\right)\left(\left[T_+^{(n)}\left(M_{v,i}^{(n),C}\eta_{v_b,i}^{(n)}\right) - M_{v,i}^{(n),C}\eta_{v_b,i}^{(n)}\right]\mathbb{1}_i^C\right)
\end{aligned}$$

$$+\sum_{i=0}^{k-1}\left(\left(T_{+}^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbb{1}_j^{(n),A}}\circ\left(T_{-}^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbb{1}_j^{(n),B}}\right)\left(\left[T_{+}^{(n)}\left(M_{v,i}^{(n),D}\right)-M_{v,i}^{(n),D}\right]\mathbb{1}_i^{(n),D}\right)\quad\text{a.s.}\tag{2.2.84}$$

Taking norms of (2.2.84) and using the isometric and commutative property of the translation operator (see claims ii) and iii) of Lemma 5.2.4) one has by the triangle inequality and the definitions of the relative volume placement and cancelation (2.1.15) that

$$\begin{aligned}
&\left\|\mathbb{E}\left[T_{+}^{(n)}\left(\eta_{v_b,k}^{(n)}\right)-\eta_{v_b,k}^{(n)}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right\|_{L^2}\\
&\quad\le\left\|T_{+}^{(n)}\left(v_{b,0}^{(n)}\right)-v_{b,0}^{(n)}\right\|_{L^2}\\
&\qquad+\sum_{i=0}^{k-1}\left\|T_{+}^{(n)}\left(\mathbb{E}\left[M_{v,i}^{(n),C}\eta_{v_b,i}^{(n)}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right)-\mathbb{E}\left[M_{v,i}^{(n),C}\eta_{v_b,i}^{(n)}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right\|_{L^2}\mathbb{E}\left[\mathbb{1}_i^{(n),C}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\\
&\qquad+\sum_{i=0}^{k-1}\left\|T_{+}^{(n)}\left(\mathbb{E}\left[M_{v,i}^{(n),D}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right)-\mathbb{E}\left[M_{v,i}^{(n),D}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right\|_{L^2}\mathbb{E}\left[\mathbb{1}_i^{(n),D}\middle|\mathcal{F}_{k-1}^{\Phi}\right]
\end{aligned}$$

$$\begin{aligned}
&\le\left\|T_{+}^{(n)}\left(v_{b,0}^{(n)}\right)-v_{b,0}^{(n)}\right\|_{L^2}\\
&\qquad+\frac{1}{n^s}\sum_{i=0}^{k-1}\mu_{\omega^C}\left\|T_{+}^{(n)}\left(f^{(n),C}\mathbb{E}\left[\eta_{v_b,i}^{(n)}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right)-f^{(n),C}\mathbb{E}\left[\eta_{v_b,i}^{(n)}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right\|_{L^2}p^{(n),C}\\
&\qquad+\frac{1}{n^s}\sum_{i=0}^{k-1}\mu_{\omega^D}\left\|T_{+}^{(n)}\left(f^{(n),D}\right)-f^{(n),D}\right\|_{L^2}p^{(n),D}
\end{aligned}$$

$$\begin{aligned}
\le\left\|T_{+}^{(n)}\left(v_{b,0}^{(n)}\right)-v_{b,0}^{(n)}\right\|_{L^2}&+\frac{1}{n^s}\sum_{i=0}^{k-1}\left\|\mathbb{E}\left[T_{+}^{(n)}\left(\eta_{v_b,i}^{(n)}\right)-\eta_{v_b,i}^{(n)}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right\|_{L^2}\\
&+\frac{1}{n^s}\sum_{i=0}^{k-1}\mu_{\omega^D}\left\|T_{+}^{(n)}\left(f^{(n),D}\right)-f^{(n),D}\right\|_{L^2}
\end{aligned}\tag{2.2.85}$$

For the the last inequality (2.2.85) we use the upper bound from part ii) of Lemma 5.2.5 in the Appendix to get:

$$\begin{aligned}
\left\|\mathbb{E}\left[T_{+}^{(n)}\left(\eta_{v_b,k}^{(n)}\right)-\eta_{v_b,k}^{(n)}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right\|_{L^2}\le\overline{K}_{T+v_{i,0}}\Delta x^{(n)}&+\frac{1}{n^s}\sum_{i=0}^{k-1}\left\|\mathbb{E}\left[T_{+}^{(n)}\left(\eta_{v_b,i}^{(n)}\right)-\eta_{v_b,i}^{(n)}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right\|_{L^2}\\
&+k\frac{\mu_{\omega^D}}{n^s}\overline{K}_{T+f^D}\Delta x^{(n)}
\end{aligned}$$

$$\leq \Delta x^{(n)}\left(\overline{K}_{T_+ v_{i,0}} + \frac{T}{\Delta t}\mu_{\omega^D}\overline{K}_{T_+ f^D}\right) + \frac{1}{n^s}\sum_{i=0}^{\lfloor\frac{n^s T}{\Delta t}\rfloor}\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,i}^{(n)}\right) - \eta_{v_b,i}^{(n)}\middle|\mathcal{F}_{i-1}^{\Phi}\right]\right\|_{L^2} \tag{2.2.86}$$

Applying the discrete Gronwall Lemma 4.2.8 to (2.2.86), we get

$$\begin{aligned}
&\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right\|_{L^2} \\
&\leq \Delta x^{(n)}\left(\overline{K}_{T_+ v_{i,0}} + \frac{T}{\Delta t}\mu_{\omega^D}\overline{K}_{T_+ f^D}\right) \\
&\quad + \frac{1}{n^s}\sum_{i=0}^{\lfloor\frac{n^s T}{\Delta t}\rfloor}\Delta x^{(n)}\left(\overline{K}_{T_+ v_{i,0}} + \frac{T}{\Delta t}\mu_{\omega^D}\overline{K}_{T_+ f^D}\right)e^{\sum_{j=i+1}^{\lfloor\frac{n^s T}{\Delta t}\rfloor}\frac{1}{n^s}} \\
&\leq \Delta x^{(n)}\underbrace{\left(\overline{K}_{T_+ v_{i,0}} + \frac{T}{\Delta t}\mu_{\omega^D}\overline{K}_{T_+ f^D}\right)\left(1 + \frac{T}{\Delta t}e^{\frac{T}{\Delta t}}\right)}_{=:\widetilde{K}^A_{\eta_{v_b}}}
\end{aligned} \tag{2.2.87}$$

and it follows that

$$\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)}\middle|\mathcal{F}_{k-1}^{\Phi}\right]\right\|_{L^2} \leq \overline{K}^A_{\eta_{v_b}}\cdot\Delta x^{(n)} = \mathcal{O}(\Delta x^{(n)}). \tag{2.2.88}$$

The analogous arguments hold for the difference using the $T_-^{(n)}$-operator with the corresponding constant $\overline{K}^B_{\eta_{v_b}}$ and for the sell side with corresponding constants $\overline{K}^E_{\eta_{v_s}}$ and $\overline{K}^F_{\eta_{v_s}}$. By symmetry, since we use the L^2-norm and the $\pm$-shifts are squared, we get the upper bound $\overline{K}_{T,\eta_v} := \overline{K}^A_{\eta_{v_b}} + \overline{K}^E_{\eta_{v_b}} = \overline{K}^B_{\eta_{v_b}} + \overline{K}^F_{\eta_{v_b}}$ of part ii).

We now show iii) using the estimates of ii). One has the following upper bound for the norm of the conditional expectation of the operator $\mathcal{D}^{(n)}_{v,k}$, where $X \in \mathbb{R}^2$ is $\mathcal{F}^{\Phi}_{k-1}$-measurable

$$\begin{aligned}
&\left\|\mathbb{E}\left[\mathcal{D}^{(n)}_{v,k}\left(X, \eta^{(n)}_{v,k}\right)\middle|\mathcal{F}^{\Phi}_{k-1}\right]\right\|_v \\
&\leq \left\|\mathbb{E}\left[\left(T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)}\right)\mathbb{1}^{(n),A}_{k-1}\middle|\mathcal{F}^{\Phi}_{k-1}\right] + \ldots + \mathbb{E}\left[M_k^{(n),D}\mathbb{1}_k^{(n),D}\middle|\mathcal{F}^{\Phi}_{k-1}\right]\right\|_{L^2} \\
&\quad + \left\|\mathbb{E}\left[\left(T_+^{(n)}\left(\eta_{v_s,k}^{(n)}\right) - \eta_{v_b,k}^{(n)}\right)\mathbb{1}^{(n),E}_{k}\middle|\mathcal{F}^{\Phi}_{k-1}\right] + \ldots + \mathbb{E}\left[M_k^{(n),H}\mathbb{1}_k^{H}\middle|\mathcal{F}^{\Phi}_{k-1}\right]\right\|_{L^2}
\end{aligned}$$

$$\leq \left\| \mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\|_{L^2} \frac{1}{n^{s-1}} + \ldots + \left\| \mathbb{E}\left[M_k^{(n),H} \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\|_{L^2}$$

$$\leq \Delta t^{(n)} \frac{\Delta x}{\Delta t} \left(\overline{K}_{\eta_{v_b}}^{A} + \overline{K}_{\eta_{v_b}}^{B} + \overline{K}_{\eta_{v_s}}^{E} + \overline{K}_{\eta_{v_s}}^{F} \right) + \frac{1}{n^s} \left(K_{M^C} + K_{M^D} + K_{M^G} + K_{M^H} \right), \tag{2.2.89}$$

where the constants $\overline{K}_{\eta_{v_b}}^{A}, \ldots, \overline{K}_{\eta_{v_s}}^{F}$ are those estimated in (2.2.88) in the proof of part ii) and the constants $K_{M^C}, \ldots, K_{M^H}$ exist by the integrability of the functions $f^{(n),C}, \ldots, f^{(n),H}$, see ii) of Lemma 5.2.5 in the Appendix. Using the estimate for $\|\eta_{v,k}^{(n)}\|_v$ of (2.2.83) and that of (2.2.89), we get

$$\frac{\left\| \mathbb{E}\left[\mathcal{D}_{v,k}^{(n)}\left(X, \eta_{v,k}^{(n)}\right) \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\|_v}{\|\eta_{v,k}^{(n)}\|_v} \leq \frac{\Delta t^{(n)} \left(\frac{\Delta x}{\Delta t} \left(\overline{K}_{\eta_{v_b}}^{A} + \ldots + \overline{K}_{\eta_{v_s}}^{F} \right) + \frac{1}{\Delta t} \left(K_{M^C} + \ldots + K_{M^H} \right) \right)}{\underline{K}_{\eta_v}}$$

and thus

$$\left\| \mathbb{E}\left[\mathcal{D}_{v,k}^{(n)}\left(X, \eta_{v,k}^{(n)}\right) \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\|_v \leq \Delta t^{(n)} \cdot L_v \cdot \|\eta_{v,k}^{(n)}\|_v \tag{2.2.90}$$

with

$$L_v := \frac{\frac{\Delta x}{\Delta t} \left(\widetilde{K}_{\eta_{v_b}}^{A} + \ldots + \widetilde{K}_{\eta_{v_s}}^{F} \right) + \frac{1}{\Delta t} \left(K_{M^C} + \ldots + K_{M^H} \right)}{\underline{K}_{\eta_v}}.$$

Thus, the inequality (2.2.9) in iii) holds and the lemma is proved. □

Proposition 2.2.10 (Convergence of $\hat{u}^{(n)}$ and $u^{(n)}$)**.** *Suppose that Assumptions 2.1.6 and 2.1.8 hold. Then we have that the process $\hat{u}^{(n)}$ defined by (2.2.59) converges uniformly to the numerical scheme $u^{(n)}$ defined by (2.2.39)-(2.2.45) in Proposition 2.2.6.*

$$\sup_{t \in [0,T]} \|\hat{u}^{(n)}(\cdot, t) - u^{(n)}(\cdot, t)\|_v = o(1) \quad \textit{a.s. as } n \to \infty. \tag{2.2.91}$$

Proof. The dynamics of the deterministic process $\hat{u}^{(n)}$ differ from those of $u^{(n)}$ by the discretization $f^{(n),I}$ of the placement functions f^I and the pre-limits $p^{(n),I}$ of the functions $p^{*,I}$. One has that by comparing terms, the linearity of the translation operators, adding and subtracting the mixed terms $\frac{p^{*,I}}{n^{s-1}} \mathbb{E}\left[M_{v,k}^{(n),I}(\hat{u}_{b,k}^{(n)}) \right]$ for $I = A, B$ and $p^{*,I} \mathbb{E}\left[M_{v,k}^{(n),I} \right]$

for $I = C, D$, we get

$$\begin{aligned}
&\left\|\hat{u_b}^{(n)}(\cdot,t) - u_b^{(n)}(\cdot,t)\right\|_{L^2} \\
&\leq \left\|\hat{u}_{b,0}^{(n)} - u_{b,0}^{(n)}\right\|_{L^2} \\
&\quad + \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t}\rfloor} \left\{ \frac{p^{*,A}\left(\hat{\gamma}(t_k^{(n)})\right)}{n^{s-1}} \left\| T_+^{(n)}\left(\hat{u}_{b,k}^{(n)} - u_{b,k}^{(n)}\right) - \left(\hat{u}_{b,k}^{(n)} - u_{b,k}^{(n)}\right)\right\|_{L^2} \right. \\
&\qquad \left. + \ldots + p^{*,D}\left(\hat{\gamma}(t_k^{(n)})\right) \frac{1}{n^s}\mu_{\omega^D} \left\| f^{(n),D} - f^D \right\|_{L^2} \right\} + o(1)
\end{aligned} \tag{2.2.92}$$

where the term of order $o(1)$ in (2.2.92) follows by the estimates

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t}\rfloor} \frac{\left| p^{*,I}\left(\gamma(t_k^{(n)})\right) - p^{(n),I}\left(\gamma(t_k^{(n)})\right)\right|}{n^{s-1}} \left\| T_\pm^{(n)}\left(\hat{u}_{b,k}^{(n)}\right) - \hat{u}_{b,k}^{(n)}\right\|_{L^2} \\
&= \mathcal{O}\left(\lfloor \frac{n^s t}{\Delta t}\rfloor\right) o\left(\frac{1}{n^s}\right) \mathcal{O}\left(\Delta x^{(n)}\right) = o(1) \quad \text{for } I = A, B,
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t}\rfloor} \left| p^{*,C}\gamma(t_k^{(n)}) - p^{(n),C}\gamma(t_k^{(n)})\right| \left\| \frac{1}{n^s}\mu_{\omega^I} f^{(n),C}\hat{u}_{b,k}^{(n)}\right\|_{L^2} \\
&= \mathcal{O}\left(\lfloor \frac{n^s t}{\Delta t}\rfloor\right) o\left(1\right) \mathcal{O}\left(\frac{1}{n^s}\right) = o(1)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t}\rfloor} \left| p^{*,D}\gamma(t_k^{(n)}) - p^{(n),D}\gamma(t_k^{(n)})\right| \left\| \frac{1}{n^s}\mu_{\omega^I} f^{(n),D}\right\|_{L^2} \\
&= \mathcal{O}\left(\lfloor \frac{n^s t}{\Delta t}\rfloor\right) o\left(1\right) \mathcal{O}\left(\frac{1}{n^s}\right) = o(1)
\end{aligned}$$

where we have used the uniform convergence of $p^{(n),I}$ to $p^{*,I}$ (see (2.1.27) of Assumption 2.1.6) and the order of the shift of the densities (see Lemma 2.2.9 ii) and the linearity of the translation operators). For the cancelation part $I = C$, we use that $\hat{u}_{b,k}$ is uniformly bounded (it may be estimated by the deterministic part of (2.2.81) in the proof of Lemma 2.2.9 ii)).

For the cancelation part in (2.2.92) we have

$$\left\| f^{(n),C}\hat{u}_{b,k}^{(n)} - f^C u_{b,k}^{(n)}\right\|_{L^2} = \left\| f^{(n),C}\hat{u}_{b,k}^{(n)} - f^C u_{b,k}^{(n)} + f^{(n),C}u_{b,k}^{(n)} - f^{(n),C}u_{b,k}^{(n)}\right\|_{L^2}$$

$$\leq \left\| f^{(n),C} \left(\hat{u}_{b,k}^{(n)} - u_{b,k}^{(n)} \right) \right\|_{L^2} + \left\| \left(f^{(n),C} - f^C \right) u_{b,k}^{(n)} \right\|_{L^2}$$

$$\leq \left\| \hat{u}_{b,k}^{(n)} - u_{b,k}^{(n)} \right\|_{L^2} + \mathcal{O}(\Delta x^{(n)}) u_{b,\max}$$

as $f^{(n),C}$ is bounded by its construction (2.1.29) and Assumption 2.1.8. The upper bound

$$u_{b,\max} := \max_{(x,t)\in[0,T]\times(-\infty,\infty)} u_b(x,t) < \infty$$

for the pointwise value of $u_{b,k}^{(n)}$ (which is a finite difference approximation) exists since the initial density of the PDE (2.2.38) is bounded by (2.1.5) and we consider the solution over a compact time interval (see the clsoed solution formula (2.2.49) in the proof of Proposition 2.2.6). By part iii) of Lemma 5.2.5 it holds that $\left\| f^{(n),C} - f^C \right\|_{L^2} = \mathcal{O}(\Delta x^{(n)})$. Thus, we have for inequality (2.2.92) above

$$\begin{aligned} \left\| \hat{u_b}^{(n)}(\cdot,t) - u_b^{(n)}(\cdot,t) \right\|_{L^2} &\leq 0 + \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \Delta t^{(n)} \hat{L}_v \left\| \hat{u}_{b,k}^{(n)} - u_{b,k}^{(n)} \right\|_{L^2} + \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \frac{1}{n^s} \mathcal{O}(\Delta x^{(n)}) + o(1) \\ &\leq \sum_{k=0}^{\lfloor \frac{n^s T}{\Delta t} \rfloor} \Delta t^{(n)} \hat{L}_v \left\| \hat{u}_{b,k}^{(n)} - u_{b,k}^{(n)} \right\|_{L^2} + \frac{T}{\Delta t} \mathcal{O}(\Delta x^{(n)}) + o(1) \end{aligned} \tag{2.2.93}$$

where the constant $\hat{L}_v$ exists by analogous arguments as in the proof of Lemma 2.2.9 ii). Applying the discrete Gronwall lemma 4.2.8 to (2.2.93) one has

$$\left\| \hat{u_b}^{(n)}(\cdot,t) - u_b^{(n)}(\cdot,t) \right\|_{L^2} \leq o(1) + \sum_{k=0}^{\lfloor \frac{n^s T}{\Delta t} \rfloor} o(1) \Delta t^{(n)} \hat{L}_v e^{\sum_{j=i+1}^{k-1} \Delta t^{(n)} \hat{L}_v} = o(1)$$

and by the analogous arguments the same holds for the sell side and the claim (2.2.91) follows. ☐

Proposition 2.2.11 (Convergence of $\tilde{u}^{(n)}$ and $\hat{u}^{(n)}$)**.** *Under Assumptions 2.1.6 and 2.1.8, it holds that the process $\tilde{u}^{(n)}$ defined in (2.2.56) converges uniformly to the process $\hat{u}^{(n)}$ defined in (2.2.57):*

$$\sup_{t\in[0,T]} \|\tilde{u}^{(n)}(\cdot,t) - \hat{u}^{(n)}(\cdot,t)\|_v = o(1) \quad a.s. \ as \ n \to \infty.$$

Proof. We study the buy and sell side separately. Recalling the operator $\mathcal{D}_{v,k}^{(n)}$ of the volume densities in (2.1.11) we have for the buy volume density component that

$$
\begin{aligned}
&\|\widetilde{u}_b^{(n)}(\cdot,t) - \hat{u}_b^{(n)}(\cdot,t)\|_{L^2} \\
&= \left\| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left(\mathbb{E}\left[\mathcal{D}_{v_b,k}^{(n)} \left(\eta_{\gamma,k}^{(n)}, \widetilde{u}_{b,k}^{(n)} \right) \Big| \mathcal{F}_{k-1}^{\Phi} \right] - \mathbb{E}\left[\mathcal{D}_{v_b,k}^{(n)} \left(\hat{\gamma}(t_k^{(n)}), \hat{u}_{b,k}^{(n)} \right) \Big| \mathcal{F}_{k-1}^{\Phi} \right] \right) \right\|_{L^2} \\
&\le \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\| \mathbb{E}\left[\mathcal{D}_{v_b,k}^{(n)} \left(\eta_{\gamma,k}^{(n)}, \widetilde{u}_{b,k}^{(n)} \right) \Big| \mathcal{F}_{k-1}^{\Phi} \right] - \mathbb{E}\left[\mathcal{D}_{v_b,k}^{(n)} \left(\hat{\gamma}(t_k^{(n)}), \hat{u}_{b,k}^{(n)} \right) \Big| \mathcal{F}_{k-1}^{\Phi} \right] \right\|_{L^2} \\
&\le \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\| \mathbb{E}\left[\mathcal{D}_{v_b,k}^{(n)} \left(\eta_{\gamma,k}^{(n)}, \widetilde{u}_{b,k}^{(n)} \right) \Big| \mathcal{F}_{k-1}^{\Phi} \right] - \mathbb{E}\left[\mathcal{D}_{v_b,k}^{(n)} \left(\eta_{\gamma,k}^{(n)}, \hat{u}_{b,k}^{(n)} \right) \Big| \mathcal{F}_{k-1}^{\Phi} \right] \right\|_{L^2} \\
&\qquad + \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\| \mathbb{E}\left[\mathcal{D}_{v_b,k}^{(n)} \left(\eta_{\gamma,k}^{(n)}, \hat{u}_{b,k}^{(n)} \right) \Big| \mathcal{F}_{k-1}^{\Phi} \right] - \mathbb{E}\left[\mathcal{D}_{v_b,k}^{(n)} \left(\hat{\gamma}(t_k^{(n)}), \hat{u}_{b,k}^{(n)} \right) \Big| \mathcal{F}_{k-1}^{\Phi} \right] \right\|_{L^2}
\end{aligned} \tag{2.2.94}
$$

$$
\le \Delta t^{(n)} \cdot \widetilde{L}_v \cdot \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \|\widetilde{u}_{b,k} - \hat{u}_{b,k}\|_{L^2} + o(1) \quad \text{a.s.} \tag{2.2.95}
$$

where the bound $\widetilde{L}_v$ for the first sum follows by iii) in Lemma 2.2.9 and the o(1) estimate for the second sum in (2.2.94) since

$$
\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\| \mathbb{E}\left[\mathcal{D}_{v_b,k}^{(n)} \left(\eta_{\gamma,k}^{(n)}, \hat{u}_{b,k}^{(n)} \right) \Big| \mathcal{F}_{k-1}^{\Phi} \right] - \mathbb{E}\left[\mathcal{D}_{v_b,k}^{(n)} \left(\hat{\gamma}(t_k^{(n)}), \hat{u}_{b,k}^{(n)} \right) \Big| \mathcal{F}_{k-1}^{\Phi} \right] \right\|_{L^2} \\
&= \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\| \left\{ \frac{p^{(n),A}\left(\eta_{\gamma,k}^{(n)}\right) - p^{(n),A}\left(\hat{\gamma}(t_k^{(n)})\right)}{n^{s-1}} \right\} \left(\hat{u}_{b,k}(\cdot + \Delta x^{(n)}) - \hat{u}_{b,k}(\cdot) \right) \right. \\
&\qquad \left. + \ldots + \left\{ p^{(n),D}\left(\eta_{\gamma,k}^{(n)}\right) - p^{(n),D}\left(\hat{\gamma}(t_k^{(n)})\right) \right\} \mathbb{E}[M_{v,k}^{(n),D}] \right\|_{L^2}
\end{aligned}
$$

$\Big\{$adding and subtracting $p^{*,I}(\eta_{\gamma,k}^{(n)})$ and $p^{*,I}(\hat{\gamma}(t_k^{(n)}))$, we have since

$\left|\eta_{\gamma,k}^{(n)} - \hat{\gamma}(t_k^{(n)})\right| = o(1)$ a.s. (see (2.2.15) in Proposition 2.2.3),

$p^{*,I}$ is Lipschitz and $p^{(n),I} \to p^{*,I}$ uniformly (Assumption 2.1.6):

$\left|p^{(n),I}\left(\eta_{\gamma,k}^{(n)}\right) - p^{*I}\left(\eta_{\gamma,k}^{(n)}\right)\right| = o(1)$ a.s. $\left|p^{*,I}\left(\hat{\gamma}(t_k^{(n)})\right) - p^{(n),I}\left(\hat{\gamma}(t_k^{(n)})\right)\right| = o(1)$ and

$\left|p^{*,I}\left(\eta_{\gamma,k}^{(n)}\right) - p^{*I}\left(\hat{\gamma}(t_k^{(n)})\right)\right| = o(1)$ a.s.

$$\left. \text{Furthermore, iii) of Lemma 2.2.9 may be applied} \right\}$$

$$\leq \lfloor \frac{n^s t}{\Delta t} \rfloor o(1) \left(\frac{1}{n^{s-1}} \cdot \mathcal{O}(\Delta x^{(n)}) + \mathcal{O}\left(\frac{1}{n^s}\right) \right) = o(1) \quad \text{a.s.}$$

Reading off (2.2.95), we have by the discrete Gronwall Lemma 4.2.8 that

$$\|\tilde{u}_b^{(n)}(\cdot,t) - \hat{u}_b^{(n)}(\cdot,t)\|_{L^2} \leq o(1) \cdot \widetilde{L}_v \cdot \mathcal{O}(\frac{1}{n^s}) n^s \frac{T}{\Delta t} e^{\widetilde{L}_v \cdot \mathcal{O}(1)} + o(1) = o(1) \tag{2.2.96}$$

since $e^{\sum_{j=k+1}^{\lfloor \frac{n^s t}{\Delta t} \rfloor - 1} \widetilde{L}_v \Delta t^{(n)}} \leq e^{\widetilde{L}_v \cdot \mathcal{O}(\frac{1}{n^s}) n^s \frac{T}{\Delta t}} = e^{\widetilde{L}_v \cdot \mathcal{O}(1)}$ which shows the desired convergence for the buy side. The analogous arguments hold for the sell side and we are done. □

Proposition 2.2.12 (Convergence of $\eta_v^{(n)}$ and $\tilde{u}^{(n)}$)**.** *Suppose that Assumptions 2.1.6 and 2.1.8 hold, then the random states volume density process $\eta_v^{(n)}$ defined in (2.2.34) converges uniformly to the process $\tilde{u}^{(n)}$ defined in (2.2.56):*

$$\sup_{t \in [0,T]} \|\eta_v^{(n)}(\cdot,t) - \tilde{u}^{(n)}(\cdot,t)\|_v = o(1) \quad a.s. \text{ as } n \to \infty.$$

Proof. Using the definition of the states of the densities $\eta_{v,k}^{(n)}$ in (2.2.35) and of the density process $\tilde{u}_k^{(n)}$ in (2.2.59) we write for the norm of their difference

$$\begin{aligned}
&\|\eta_v^{(n)}(\cdot,t) - \widetilde{u}^{(n)}(\cdot,t)\|_v \\
&\leq \| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \Big(\mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)}) \Big| \mathcal{F}_{k-1}^{\Phi}] - \mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)}, \widetilde{u}_k^{(n)}) \Big| \mathcal{F}_{k-1}^{\Phi}] \Big) \|_v \\
&\quad + \| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \Big(\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)}) - \mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)}) \Big| \mathcal{F}_{k-1}^{\Phi}] \Big) \|_v \\
&\leq \Delta t^{(n)} \cdot L_v \cdot \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\| \eta_{v,k}^{(n)} - \widetilde{u}_k^{(n)} \right\|_v \\
&\quad + \| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \Big(\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)}) - \mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)}) \Big| \mathcal{F}_{k-1}^{\Phi}] \Big) \|_v
\end{aligned} \tag{2.2.97}$$

where the last inequality follows by the bound of the conditional expectation of the random state operator in (2.2.79) of Lemma 2.2.9 (the estimate (2.2.90) also holds for $\tilde{u}_k^{(n)}$).

We now show that the second term of (2.2.97) converges to 0 a.s. For the buy side we have by the triangle inequality that

$$\| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \Big(\mathcal{D}^{(n)}_{v_b,k}(\eta^{(n)}_{\gamma,k}, \eta^{(n)}_{v_b,k}) - \mathbb{E}[\mathcal{D}^{(n)}_{v_b,k}(\eta^{(n)}_{\gamma,k}, \eta^{(n)}_{v_b,k}) \Big| \mathcal{F}^{\Phi}_{k-1}] \Big) \|_{L^2}$$

$$\begin{aligned}
&\leq \left\| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\{ M^{(n),A}_{v,k} \left(\eta^{(n)}_{v_b,k} \right) \mathbb{1}^{(n),A}_k \left(\eta^{(n)}_{\gamma,k} \right) - \mathbb{E} \left[M^{(n),A}_{v,k} \left(\eta^{(n)}_{v_b,k} \right) \mathbb{1}^{(n),A}_k \left(\eta^{(n)}_{\gamma,k} \right) \Big| \mathcal{F}^{\Phi}_{k-1} \right] \right\} \right\|_{L^2} \\
&+ \left\| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\{ M^{(n),B}_{v,k} \left(\eta^{(n)}_{v_b,k} \right) \mathbb{1}^{(n),B}_k \left(\eta^{(n)}_{\gamma,k} \right) - \mathbb{E} \left[M^{(n),B}_{v,k} \left(\eta^{(n)}_{v_b,k} \right) \mathbb{1}^{(n),B}_k \left(\eta^{(n)}_{\gamma,k} \right) \Big| \mathcal{F}^{\Phi}_{k-1} \right] \right\} \right\|_{L^2} \\
&+ \Bigg\| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \Bigg\{ M^{(n),C}_{v,k} \eta^{(n)}_{v_b,k} \mathbb{1}^{(n),C}_k \left(\eta^{(n)}_{\gamma,k} \right) + M^{(n),D}_{v,k} \mathbb{1}^{(n),D}_k \left(\eta^{(n)}_{\gamma,k} \right) \\
&\qquad - \mathbb{E} \left[M^{(n),C}_{v,k} \eta^{(n)}_{v_b,k} \mathbb{1}^{(n),C}_k \left(\eta^{(n)}_{\gamma,k} \right) + M^{(n),D}_{v,k} \mathbb{1}^{(n),D}_k \left(\eta^{(n)}_{\gamma,k} \right) \right] \Big| \mathcal{F}^{\Phi}_{k-1} \Bigg\} \Bigg\|_{L^2}
\end{aligned} \tag{2.2.98}$$

To see that the third term in (2.2.98) diminishes, we argue as in the proof of Lemma 2.6 ii): using the step function property of the terms and showing convergence for the heights of the steps over each price interval $[x^{(n)}_j, x^{(n)}_{j+1})$. By Remark 2.1.7 it follows that these heights are an $\mathcal{F}_k$-martingale difference sequence. As we have shown the uniform bound of $\eta^{(n)}_{v_b,k}$ in Lemma 2.2.9 i), the corresponding uniform bounds of the second moment (2.2.72) holds for the heights when cancelation is included. Thus, we accordingly have as in Lemma 2.2.7 ii) that

$$\begin{aligned}
&\Bigg\| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \Bigg\{ M^{(n),C}_{v,k} \eta^{(n)}_{v_b,k} \mathbb{1}^{(n),C}_k \left(\eta^{(n)}_{\gamma,k} \right) + M^{(n),D}_{v,k} \mathbb{1}^{(n),D}_k \left(\eta^{(n)}_{\gamma,k} \right) \\
&\qquad - \mathbb{E} \left[M^{(n),C}_{v,k} \eta^{(n)}_{v_b,k} \mathbb{1}^{(n),C}_k \left(\eta^{(n)}_{\gamma,k} \right) + M^{(n),D}_{v,k} \mathbb{1}^{(n),D}_k \left(\eta^{(n)}_{\gamma,k} \right) \Big| \mathcal{F}^{\Phi}_{k-1} \right] \Bigg\} \Bigg\|_{L^2} \\
&= \mathcal{O}\left(n^{\frac{-\delta}{2}(s-1)} \right).
\end{aligned} \tag{2.2.99}$$

It follows that the third term in (2.2.98) is of order $\mathcal{O}\left(\frac{1}{n}\right)$.

The first term of (2.2.98) may be bounded as follows. The number of A-events are a.s. of order $\mathcal{O}(n)$ by Lemma 2.2.7 i) and we thus have

$$\begin{aligned}&\left\| \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\{ M_{v,k}^{(n),A}\left(\eta_{v_b,k}^{(n)}\right) \mathbb{1}_k^{(n),A}\left(\eta_{\gamma,k}^{(n)}\right) - \mathbb{E}\left[M_{v,k}^{(n),A}\left(\eta_{v_b,k}^{(n)}\right) \mathbb{1}_k^{(n),A}\left(\eta_{\gamma,k}^{(n)}\right) \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\} \right\|_{L^2} \\ &= \left\| \sum_{k=0}^{\mathcal{O}(n)} \left\{ T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} - \mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\} \right\|_{L^2} \quad \text{i.o.} \qquad (2.2.100)\end{aligned}$$

We proceed to estimate the L^2-norm of $T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)}$ i.e. the shift due to a market order after k events by using the induction formula (2.2.80). The number of C *or* D-events are i.o. of order $\mathcal{O}(k)$, given that k events have occurred, by arguing analogously as in the proof of Lemma 2.2.7 i) for the price changing events A/B and E/F. Thus, using relation (2.2.84) for the price shifts, taking norms, adding and subtracting the conditional expectation and using the isometric property of the translation operators we get

$$\begin{aligned}&\left\| T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} \right\|_{L^2} \\ &\leq \left\| T_+^{(n)}\left(v_{b,0}^{(n)}\right) - v_{b,0}^{(n)} \right\| \\ &+ \Bigg\| \sum_{i=0}^{\mathcal{O}(k)} T_+^{(n)}\left(M_{v,i}^{(n),C} \eta_{v_b,i}^{(n)} \mathbb{1}_i^C + M_{v,i}^{(n),D} \mathbb{1}_i^D \right) \\ &\qquad - T_+^{(n)}\left(\mathbb{E}\left[M_{v,i}^{(n),C} \eta_{v_b,i}^{(n)} \mathbb{1}_i^C + M_{v,i}^{(n),D} \mathbb{1}_i^D \middle| \mathcal{F}_{i-1}^{\Phi} \right] \right) \\ &\qquad - \left(M_{v,i}^{(n),C} \eta_{v_b,i}^{(n)} \mathbb{1}_i^C + M_{v,i}^{(n),D} \mathbb{1}_i^D - \mathbb{E}\left[M_{v,i}^{(n),C} \eta_{v_b,i}^{(n)} \mathbb{1}_i^C + M_{v,i}^{(n),D} \mathbb{1}_i^D \middle| \mathcal{F}_{i-1}^{\Phi} \right] \right) \Bigg\|_{L^2} \\ &+ \Bigg\| \sum_{i=0}^{\mathcal{O}(k)} T_+^{(n)}\left(\mathbb{E}\left[M_{v,i}^{(n),C} \eta_{v_b,i}^{(n)} \mathbb{1}_i^C + M_{v,i}^{(n),D} \mathbb{1}_i^D \middle| \mathcal{F}_{i-1}^{\Phi} \right] \right) \\ &\qquad - \mathbb{E}\left[M_{v,i}^{(n),C} \eta_{v_b,i}^{(n)} \mathbb{1}_i^C + M_{v,i}^{(n),D} \mathbb{1}_i^D \middle| \mathcal{F}_{i-1}^{\Phi} \right] \Bigg\|_{L^2} \quad \text{i.o.} \\ &\leq \mathcal{O}\left(\Delta x^{(n)}\right) + \mathcal{O}\left(k^{-\frac{\delta}{2}\left(1-(1/s)\right)}\right) + \mathcal{O}(k) \frac{1}{n^s} \mathcal{O}\left(\Delta x^{(n)}\right) \\ &\overset{k \leq \lfloor \frac{n^s t}{\Delta t} \rfloor}{=} \mathcal{O}\left(\Delta x^{(n)}\right) \quad \text{i.o.} \qquad (2.2.101)\end{aligned}$$

The first rate follows from Lemma 5.2.5 ii), the second rate is just an application of (2.2.99) and the third rate follows from Lemma 2.2.9 ii) and the linearity of the shift operator. We now claim that

$$\begin{aligned}
&\left\| \sum_{k=0}^{\mathcal{O}(n)} \left\{ T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} - \mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\} \right\|_{L^2} \\
&= \left\| \frac{1}{n} \sum_{k=0}^{\mathcal{O}(n)} \underbrace{n\left\{ T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} - \mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\}}_{=:X_{b,k}^{(n)}\left(\eta_{v_b,k}^{(n)}\right)} \right\|_{L^2} = o(1) \quad a.s.
\end{aligned} \tag{2.2.102}$$

by applying the SLLN Theorem 4.2.4 for Banach spaces. We first note that the sequence $\left\{X_{b,k}^{(n)}(\eta_{v_b,k}^{(n)})\right\}_{k\geq 0}$ is an $\mathcal{F}_k$-martingale difference sequence for all n. All that is left to show is that the summability condition (4.2.1) holds uniformly in n, i.e that

$$\sum_{k=1}^{\infty} \frac{\mathbb{E}\left[\|X_{b,k}^{(n)}(\eta_{v_b,k}^{(n)})\|_{L^2}^2\right]}{k^2} < \infty \quad \text{uniformly in } n. \tag{2.2.103}$$

We calculate

$$\begin{aligned}
\mathbb{E}\left[\|X_{b,k}^{(n)}(\eta_{v_b,k}^{(n)})\|_{L^2}^2\right] &= n^2 \mathbb{E}\left[\left\| T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} - \mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right) - \eta_{v_b,k}^{(n)} \middle| \mathcal{F}_{k-1}^{\Phi} \right] \right\|_{L^2}^2 \right] \\
&\leq n^2 \left(\mathcal{O}\left(\Delta x^{(n)}\right) \right)^2 = \mathcal{O}(1)
\end{aligned}$$

for all $k \leq \lfloor \frac{n^s T}{\Delta t} \rfloor \leq \frac{n^s t}{\Delta T}$ and all $n > 1$ by (2.2.101) and Lemma 2.2.9 ii) which verifies (2.2.103). Thus, the noise from the market orders diminishes a.s. by (2.2.102). An analogous bound holds for event B and this means that the entire noise of the buy side (2.2.98) is of order $o(1)$ a.s. As we may argue in the same way for the sell side we conclude that the combined noise of the volume densities is of order $o(1)$ a.s. as $n \to \infty$. Recalling (2.2.97), we have

$$\|\eta_{v_b}^{(n)}(\cdot, t) - \tilde{u}_b^{(n)}(\cdot, t)\|_{L^2} \leq \Delta t^{(n)} L \sum_{k=0}^{\lfloor \frac{n^s t}{\Delta t} \rfloor} \left\| \eta_{v,k}^{(n)} - \tilde{u}_k^{(n)} \right\|_{L^2} + o(1) \quad \text{a.s.}$$

By the discrete Gronwall Lemma 4.2.8, we have that

$$\|\eta_v^{(n)}(\cdot, t) - \tilde{u}^{(n)}(\cdot, t)\|_v \leq o(1) \cdot L \cdot \mathcal{O}(\frac{1}{n^s}) n^s \frac{T}{\Delta t} e^{L \cdot \mathcal{O}(1)} + o(1) = o(1) \tag{2.2.104}$$

since $e^{\sum_{j=k+1}^{\lfloor \frac{n^s t}{\Delta t} \rfloor - 1} L \Delta t^{(n)}} \leq e^{L \cdot \mathcal{O}(\frac{1}{n^s}) n^s T} = e^{L \cdot \mathcal{O}(1)}$ which shows the desired convergence. □

Propositions 2.2.6-2.2.12 show that all the terms of (2.2.60) converge uniformly a.s. and we conclude that the states of the volume densities indeed converge

$$\sup_{t\in[0,T]} \|\eta_v^{(n)}(\cdot,t) - u(\cdot,t)\|_v \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty. \tag{2.2.105}$$

This fact along with convergence of the prices and the inter arrival time process in Proposition 2.2.3 yield that

$$\left\|\eta^{(n)}\left(\cdot,\mu^{(n)}(t) - \Delta t^{(n)}\right) - \eta(\cdot,\mu(t))\right\|_E = \left\|S^{(n)}(\cdot,t) - s(\cdot,t)\right\|_E \to 0 \quad \text{a.s. as } n \to \infty,$$

uniformly for $t \in [0,T]$ where

$$\frac{\mathrm{d}}{\mathrm{d}t}s(t) = \frac{\mathrm{d}}{\mathrm{d}t}\eta(\mu(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\eta(\mu(t)) \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(y^{-1}(t)\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\eta(\mu(t))}{\frac{\mathrm{d}}{\mathrm{d}t}y(y^{-1}(t))} = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\eta(\mu(t))}{m^*(\gamma(t))},$$

we have $\eta = (\eta_\gamma, \eta_v)' = (\hat{\gamma}, u)'$ so the limiting model s solves (2.1.30) and (2.1.31).

We assumed that the inter arrival times are a.s. positive and that $m^*(\gamma)$ is globally Lipschitz in Assumption 2.1.6. $b, a \in C^2([0,T],\mathbb{R})$ solve (2.1.30) and $v_b, v_s \in C^{2,2}(\mathbb{R} \times [0,T], \mathbb{R}_{>0})$ solve (2.1.31) by the arguments in the beginning of the proofs of Lemma 2.2.1 and Proposition 2.2.6, respectively. Also, $m^* > 0$ over $[0,T]$ by assumption (2.1.25) and γ is C^2 so we are done proving the main result.

2.3 A Weak Law of Large Numbers

In Horst and Paulsen [45] a weak law of large numbers was proved for an order book model of similar type as in the previous section of this chapter. The scaling has some different features, however, which advantageously may be used to prove weak convergence by utilizing a result by Pisier for martingale difference sequences in Banach space (Lemma 5.2.7).

2.3.1 A Sequence of Discrete Order Book Models

In this section, we introduce a sequence of order book models for which we establish a scaling limit when the price tick and impact of a single order on the state of the book tend to zero, while the rate of order arrivals tends to infinity.

The sequence of models is indexed by $n \in \mathbb{N}$. We assume that the set of price levels at which orders can be submitted in the n:th models is $\{x_j^{(n)}\}_{i\in\mathbb{Z}}$ where $\mathbb{Z}$ denotes the one-dimensional integer lattice.[7] We put $x_j^{(n)} := j \cdot \Delta x^{(n)}$ for $j \in \mathbb{Z}$ where $\Delta x^{(n)}$ is the

[7] The assumption that there is no minimum price is made for analytical convenience and can easily be relaxed.

tick size in the n:th model.

The *state* of the book changes due to incoming order flow and cancelations of standing volume. The state after $k \in \mathbb{N}$ such *events* will be described by a random variable $S_k^{(n)}$ taking values in a suitable *state space* E. In the n:th model, the k:th event occurs at a random point in time $\tau_k^{(n)}$. The time between two consecutive events will be tending to zero sufficiently fast as $n \to \infty$. The state and time dynamics will be defined, respectively, as

$$S_0^{(n)} := s_0^{(n)}, \quad S_{k+1}^{(n)} := S_k^{(n)} + \mathcal{D}_k^{(n)}(S_k^{(n)}) \tag{2.3.1}$$

and

$$\tau_0^{(n)} := 0, \quad \tau_{k+1}^{(n)} := \tau_k^{(n)} + \mathcal{C}_k^{(n)}(S_k^{(n)}). \tag{2.3.2}$$

Here $s_0^{(n)} \in E$ is a deterministic initial state, and $\mathcal{D}_k^{(n)}(S_k^{(n)}) : E \to E$ and $\mathcal{C}_k^{(n)}(S_k^{(n)}) : E \to [0, \infty)$ are random operators that will be introduced below. The conditional expected increment of the state sequence, given the current state[8] $S_k^{(n)}$, will be denoted $\mathbb{E}\left[\mathcal{D}_k^{(n)}(S_k^{(n)})\right]$; the unconditional increment will be denoted $\mathbb{E}\left[\mathcal{D}_k^{(n)}\right]$.

In the sequel we specify the dynamics of our order book models. Throughout all random variables will be defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The initial state.

The initial state of the book in the n:th model is given by a pair $(B_0^{(n)}, A_0^{(n)})$ of best bid and ask prices together with the standing buy and sell limit order volumes at the various price levels. It will be convenient to the identify standing volumes with step functions

$$v_{b,0}^{(n)}(x) := \sum_{j=0}^{\infty} v_{b,0}^{(n),j} \mathbf{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(x), \quad v_{s,0}^{(n)}(x) := \sum_{j=0}^{\infty} v_{s,0}^{(n),j} \mathbf{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(x) \quad (x \geq 0)$$

that specify the liquidity available for buying and selling *relative* to the best bid and ask price. The liquidity available for buying (sell side of the book) $j \in \mathbb{N}_0$ ticks *above* the best ask price at the price level $x_{A_0^{(n)}+j}^{(n)}$ is

$$\int_{x_j^{(n)}}^{x_{j+1}^{(n)}} v_{s,0}^{(n)}(x)dx = v_{s,0}^{(n),j} \cdot \Delta x^{(n)}.$$

[8]Since we will assume that the dynamics are driven by the best bid and ask prices, it will be sufficient to condition on the current best bid and ask prices. In the first part of the chapter, we similarly conditioned on the event sub-σ-algebra $\mathcal{F}^{\Phi}$ (2.1.24).

The volume available for selling (buy side of the book[9]) $l \in \mathbb{N}_0$ ticks *below* the best bid price at the price level $x^{(n)}_{B_0^{(n)}-l}$ is

$$\int_{x_l^{(n)}}^{x_{l+1}^{(n)}} v_{b,0}^{(n)}(x)dx = v_{b,0}^{(n),l} \cdot \Delta x^{(n)}.$$

In order to conveniently model placements of limit orders into the spread, we extend $v_{b,0}^{(n)}$ and $v_{s,0}^{(n)}$ to the negative half-line. The collection of volumes standing at negative distances from the best bid/ask price is referred to as the *shadow book*. The shadow book will undergo the same dynamics as the standing volume ("visible book"). At any point in time it specifies the volumes that will be placed into the spread should such an event occur next.

Definition 2.3.1. *In the n:th model the initial state of the book is given by a quadruple*

$$S_0^{(n)}(\cdot) = \left(B_0^{(n)}, A_0^{(n)}, v_{b,0}^{(n)}(\cdot), v_{s,0}^{(n)}(\cdot)\right)'$$

where $B_0^{(n)} \leq A_0^{(n)}$ are the best bid/ask price and the step functions $v_{b,0}^{(n)}, v_{s,0}^{(n)} : \mathbb{R} \to [0,\infty)$ are to be interpreted as follows:

$$v_{b,0}^{(n)}(x)\ \left[v_{s,0}^{(n)}(x)\right] := \begin{cases} \textit{standing buy [sell] limit order volume density at price distance } x \\ \textit{below [above] the best bid [ask] price, for } x \geq 0 \textit{ (visible book)} \\[1ex] \textit{potential buy [sell] limit order volume density at price distance } x \\ \textit{above [below] the best bid [ask] price, for } x < 0 \textit{ (shadow book).} \end{cases} \tag{2.3.3}$$

With this, we are now ready to state the assumptions on the initial states, which differ somewhat to those used above. In particular, we assume that the initial volume density functions vanish outside a compact price interval.[10]

Assumption 2.3.2 (Convergence of initial states)**.** *The initial volume density functions vanish outside a compact interval $[-M, M]$ for some $M > 0$. Moreover, there exists non-negative bounded and continuously differentiable functions $v_{r,0} \in L^2$ $(r \in \{b, s\})$ with bounded derivatives such that as $n \to \infty$*

$$||v_{r,0}^{(n)} - v_{r,0}||_{L^2} = o(1)$$

[9]Notice that the liquidity available for buying is captured by the sell side of the book and vice versa.

[10]This assumption considerably simplifies some of the analysis that follows. But it can easily be dropped.

as well as

$$\|v_{r,0}^{(n)}(\cdot \pm \Delta x^{(n)}) - v_{r,0}^{(n)}(\cdot)\|_{L^2} = \mathcal{O}(\Delta x^{(n)}).$$

Here, $\|\cdot\|_{L^2}$ *denotes the* L^2*-norm on* $\mathbb{R}$ *with respect to Lebesgue measure. Moreover,*

$$\lim_{n\to\infty}(B_0^{(n)}, A_0^{(n)}) = (B_0, A_0).$$

The first condition on the volume density functions is intuitive. The second condition will become clear later; it will be used to bound the impact of market orders and limit orders placed into the spread on the state of the book. Under mild smoothness conditions it is not difficult to show that both assumptions are satisfied, if the initial volume density functions originate from common benchmark functions, see Lemma 5.2.5 of the Appendix.

Event types.

There are eight events - labeled **A, ..., H** - that change the state of the book. The events **A, ..., D** effect the buy side of the book:

$$\mathbf{A} := \{\text{market sell order}\} \qquad \mathbf{B} := \{\text{buy limit order placed in the spread}\}$$
$$\mathbf{C} := \{\text{cancelation of buy volume}\} \qquad \mathbf{D} := \{\text{buy limit order not placed in spread}\}$$

The remaining four events effect the sell side of the book:

$$\mathbf{E} := \{\text{market buy order}\} \qquad \mathbf{F} := \{\text{sell limit order placed in the spread}\}$$
$$\mathbf{G} := \{\text{cancelation of sell volume}\} \qquad \mathbf{H} := \{\text{sell limit order not placed in the spread}\}.$$

We will again describe the state dynamics of the n:th model by a stochastic process $\{S_k^{(n)}\}_{k\in\mathbb{N}}$ that takes values in the Hilbert space

$$E := \mathbb{R} \times \mathbb{R} \times L^2 \times L^2.$$

The first two components of the vector $S_k^{(n)}$ stand for the best bid and ask price after k events; the third and fourth component refer to the buy and sell volume density functions relative to the best bid and ask price, respectively (visible and shadow book). We define a norm on E by

$$\|\alpha\|_E := |\alpha_1| + |\alpha_2| + \|\alpha_3\|_{L^2} + \|\alpha_4\|_{L^2}, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in E. \tag{2.3.4}$$

In the sequel we specify how different events change the state of the book and how order arrival times and sizes scale with the parameter $n \in \mathbb{N}$.

Active orders.

Market orders and placements of limit orders in the spread lead to price changes[11]. With a slight abuse of terminology we refer to these order types as *active orders*.

For convenience, we assume that market orders match precisely against the standing volume at the best prices and that limit orders placed in the spread improve prices by one tick. The assumptions that market orders decrease (increase) the best bid (ask) price by one tick while limit orders placed in the spread decrease (increase) prices by the same amount have been made in the literature before and can be generalized without too much effort.

If the k:th event is a sell market order (Event $\mathbf{A}$), then the relative buy volume density shifts one price tick to the left, the best bid price decreases by one tick and the relative sell volume density and the best ask price remain unchanged. Since the relative volume density functions are defined on the whole real line, the transition operators

$$T_+^{(n)}(v)(\cdot) = v(\cdot + \Delta x^{(n)}), \quad T_-^{(n)}(v)(\cdot) = v(\cdot - \Delta x^{(n)})$$

are well defined and one has that

$$v_{b,k+1}^{(n)}(\cdot) = T_+^{(n)}\left(v_{b,k}^{(n)}\right)(\cdot), \quad v_{s,k+1}^{(n)}(\cdot) = v_{s,k}^{(n)}(\cdot)$$

and

$$B_{k+1}^{(n)} = B_k^{(n)} - \Delta x^{(n)}, \quad A_{k+1}^{(n)} = A_k^{(n)}.$$

The placement of orders into the spread will be modeled using the shadow book. If the k:th event is a buy limit order placement in the spread (Event $\mathbf{B}$), the relative buy volume density shifts one price tick to the right, the best bid price increases by one tick and the relative buy volume density and the best ask price remain unchanged:

$$v_{b,k+1}^{(n)}(\cdot) = T_-^{(n)}\left(v_{b,k}^{(n)}\right)(\cdot), \quad v_{s,k+1}^{(n)}(\cdot) = v_{s,k}^{(n)}(\cdot)$$

and

$$B_{k+1}^{(n)} = B_k^{(n)} + \Delta x^{(n)}, \quad A_{k+1}^{(n)} = A_k^{(n)}.$$

Passive orders.

Limit order placements outside the spread and cancelations of standing volume do not change prices. With the same minor abuse of terminology as before, we refer to these order types as *passive orders*.

[11] A market order that does not lead to a price change can be viewed as a cancelation of standing volume at the best bid/ask price.

We assume that cancelations of buy volume (Event **C**) occur for random *proportions* of the standing volume at random price levels while limit buy order placements outside the spread (Event **D**) occur for random *volumes* at random price levels. The submission and cancelation price levels are chosen relative to the best bid price. To guarantee convergence of the dynamics of volumes to the solution of a PDE, volumes placed and proportions canceled will be scaled in the following way, which entail slightly stronger assumptions than were used in the first part of the chapter.

Assumption 2.3.3. *For each $k \in \mathbb{N}$ there exist random variables ω_k^C, ω_k^D taking values in $(0,1)$, respectively $[0, M]$ for some $M > 0$ and random variables π_k^C, π_k^D taking values in $[-M, M]$ such that, if the k:th event is a limit buy order cancelation/placement, then it occurs at the price level $x^{(n)}_{B_k^{(n)}-j}$ $(j \in \mathbb{Z})$ for which*

$$\pi_k^{C,D} \in [x_j^{(n)}, x_{j+1}^{(n)}).$$

The volume canceled, respectively, placed is

$$\omega_k^C \cdot \Delta v^{(n)} \cdot v_{b,k}^{(n)}(\pi_k^C) \quad \textit{respectively} \quad \omega_k^D \cdot \Delta v^{(n)}.$$

Here $v_{b,k}^{(n)}(\pi_k^C)$ is the value of the volume density functions at the cancelation price level and $\Delta v^{(n)}$ is a scaling parameter that describes the impact of an individual limit order arrival (cancelation) on the state of the order book.

As before, by construction, buy order placements and cancelations take place at random distances (positive or negative) from the best bid price. Since $|\pi_k^{C,D}| \leq M$ no orders are placed/canceled more than $M/\Delta x^{(n)}$ ticks into the book. Volume changes take place in the visible or the shadow book, depending on the sign of π_k^I. If $\pi_k^I \geq 0$, then the visible book changes; if $\pi_k^I < 0$ the placement/cancelation takes place in the shadow book and the impact of the event on the state of the visible book (i.e. at price levels *above* the best bid) will be felt only after a price increase.

Second, the liquidity available for buying $j \in \mathbb{N}$ ticks into the book after k events is given by by the integral of the volume density function over the interval $[x_j^{(n)}, x_{j+1}^{(n)})$, i.e. equals $v_{b,k}^{(n)}(x_j^{(n)}) \cdot \Delta x^{(n)}$. When a cancelation occurs at this price level, then the new volume is $(\Delta x^{(n)} - \omega_k^C \cdot \Delta v^{(n)}) \cdot v_{b,k}^{(n)}(x_j^n) = (1 - \omega_k^C \frac{\Delta v^{(n)}}{\Delta x^{(n)}}) \cdot v_{b,k}^{(n)}(x_j^n) \cdot \Delta x^{(n)}$. Hence cancelations are proportional to standing volumes. On the level of the volume density functions, Assumption 2.3.3 implies that

$$v_{b,k+1}^{(n)}(\cdot) = v_{b,k}^{(n)}(\cdot) - \frac{\Delta v^{(n)}}{\Delta x^{(n)}} \cdot M_k^{(n),C}(\cdot) \cdot v_{b,k}^{(n)}(\cdot), \quad \text{where}$$
$$M_k^{(n),C}(x) := \omega_k^C \sum_{j=-\infty}^{\infty} \mathbf{1}_{\{\pi_k^C \in [x_j^{(n)} x_{j+1}^{(n)})\}}(x)..$$

Volume placements are additive. If the order a limit buy order, then the volume density function changes according to

$$v_{b,k+1}^{(n)}(\cdot) = v_{b,k}^{(n)}(\cdot) + \frac{\Delta v^{(n)}}{\Delta x^{(n)}} \cdot M_k^{(n),D}(\cdot), \quad \text{where}$$
$$M_k^{(n),D}(x) := \omega_k^D \sum_{j=-\infty}^{\infty} \mathbf{1}_{\{\pi_k^D \in [x_j^{(n)}, x_{j+1}^{(n)})\}}(x). \tag{2.3.5}$$

In either case, the bid/ask price and standing sell side volume of the book remain unchanged:

$$v_{s,k+1}^{(n)}(\cdot) = v_{s,k}^{(n)}(\cdot), \quad B_{k+1}^{(n)} = B_k^{(n)}, \quad A_{k+1}^{(n)} = A_k^{(n)}.$$

Similar considerations apply to the sell side with respective random quantities ω_k^G, ω_k^H and π_k^G, π_k^H.

Assumption 2.3.4. *For $I \in \{\mathbf{C}, \mathbf{D}, \mathbf{G}, \mathbf{H}\}$ the sequences $\{\omega_k^I\}_{k\in\mathbb{N}}$ and $\{\pi_k^I\}_{k\in\mathbb{N}}$ are independent sequences of i.i.d. random variables. Moreover, the (scaled) expected changes in the heights of the density volume functions, due to cancelations and order placements*

$$f^{(n),I}(\cdot) := \frac{1}{\Delta x^{(n)}} \mathbb{E}\left[\omega_k^I \sum_{j=-\infty}^{\infty} \mathbf{1}_{\{\pi_k^I \in [x_j^{(n)}, x_{j+1}^{(n)})\}}(\cdot)\right], \quad I \in \{\mathbf{C}, \mathbf{D}, \mathbf{G}, \mathbf{H}\}$$

belong to $L^2$and there exists bounded continuously differentiable functions $f^I \in L^2$ with uniformly bounded derivatives such that as $n \to \infty$

$$\|f^{(n),I} - f^I\|_{L^2} = o(1) \quad \textit{and} \quad \|T_{\pm}^{(n)} \circ f^{(n),I} - f^{(n),I}\|_{L^2} = \mathcal{O}(\Delta x^{(n)}). \tag{2.3.6}$$

Event times.

The dynamics of event times is specified in terms of a sequence of inter arrival times whose distributions may depend on prevailing best bid and ask prices.

Assumption 2.3.5. *Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a sequence of non-negative random variables that* *are conditionally independent, given the current best bid and ask price:*

$$\mathbb{P}\left(\varphi_k \leq t | S_k^{(n)}\right) = \mathbb{P}\left(\varphi_k \leq t | B_k^{(n)}, A_k^{(n)}\right).$$

In the sequel we write $\varphi(A_k, B_k)$ for φ_k to indicate the dependence of the distribution of φ_k on the best bid and ask price. Similar notation will be applied to other random variables whenever convenient.

In the n:th model, we scale the time by a factor $\Delta t^{(n)}$. More precisely, we assume that the dynamics of the event times in the n:the model is given by:

$$\tau_{k+1}^{(n)} = \tau_k^{(n)} + \mathcal{C}_k^{(n)}(B_k^{(n)}, A_k^{(n)}), \text{ where } \quad \mathcal{C}_k^{(n)}(B_k^{(n)}, A_k^{(n)}) := \varphi(B_k^{(n)}, A_k^{(n)}) \cdot \Delta t^{(n)}. \quad (2.3.7)$$

Event types.

Event types are described in terms of a sequence of random *event indicator variables* $\{\phi_k\}$ taking values in the set $\{\mathbf{A}, ..., \mathbf{H}\}$. We assume that the random variables

$$\phi_k = \phi_k(B_k^{(n)}, A_k^{(n)}) \quad (k \in \mathbb{N})$$

are conditionally independent, given the prevailing best bid and ask price and that their conditional probabilities

$$p_k^{(n)} = p^{(n),I}(B_k^{(n)}, A_k^{(n)}) := \mathbb{P}\left(\phi_k = I | S_k^{(n)}\right)$$

satisfy the following condition.

Assumption 2.3.6. *There are bounded continuous functions with bounded gradients* $p^I : \mathbb{R}^2 \to [0,1]$ *and a scaling parameter* $\Delta p^{(n)} \to 0$ *such that*

$$\begin{aligned} p^{(n),I}(\cdot,\cdot) &= \Delta p^{(n)} \cdot p^I(\cdot,\cdot) && \textit{for} \quad I = A, B, E, F \\ p^{(n),I}(\cdot,\cdot) &= (1 - \Delta p^{(n)}) \cdot p^I(\cdot,\cdot) && \textit{for} \quad I = C, D, G, H \\ p^A + p^B + p^E + p^F &= 1 \\ p^C + p^D + p^G + p^H &= 1 \end{aligned}$$

The preceding assumption implies that an event is an active order with probability $\Delta p^{(n)}$ and a passive order with probability $1 - \Delta p^{(n)}$, independently of the state of the book. Conditioned on the order being active or passive, it is of type I with a probability $p^I(\cdot,\cdot)$ that depends on the current best bid and ask prices. We allow the above probabilities to be zero in order to (i) account for the fact that no price improvements can take place when $B_k^{(n)} = A_k^{(n)}$ and (ii) to avoid depletion of the order book.[12]

The expected impact of each active order on the state of the book will be of order $\Delta x^{(n)}$; that of a passive order of order $\Delta v^{(n)}$. Because active orders arrive at a rate that is $\Delta p^{(n)}$-times slower than that of passive orders, the relative average impact of active to passive orders on the state of the book will be of the order $\frac{\Delta p^{(n)} \Delta x^{(n)}}{\Delta v^{(n)}}$. Our scaling limit requires to equilibrate the impact of active and passive orders. In order to guarantee that there will be no fluctuations in the standing volumes in the limit as $n \to \infty$ we also need a minimum relative frequency of passive order arrivals. This motivates the following assumption.

Assumption 2.3.7. *The scaling constants $\Delta p^{(n)}$, $\Delta x^{(n)}$, $\Delta v^{(n)}$ and $\Delta t^{(n)}$ are such that:*

$$\lim_{n\to\infty} \frac{\Delta x^{(n)} \cdot \Delta p^{(n)}}{\Delta v^{(n)}} = c_0, \quad \lim_{n\to\infty} \frac{\Delta v^{(n)}}{\Delta t^{(n)}} = c_1, \quad \text{and} \quad \frac{\Delta p^{(n)}}{\left(\Delta t^{(n)}\right)^{\alpha}} = \mathcal{O}(1)$$

for some $\alpha > \frac{1}{2}$ and constants $c_0, c_1 > 0$.[13]

Active order times.

The previous two assumptions introduce two different time scales for order arrivals: a fast time scale for passive order arrivals, and a comparably slow time scale for active order arrivals. Inter arrival times between passive orders are of the order $\Delta t^{(n)}$ while inter arrival times between active orders are of the order $\Delta x^{(n)}$. In order to see this, let us denote by $\sigma_k^{(n)}$ the arrival time of the k:th active order. The *number* $r_{k+1}^{(n)}$ of events one needs to wait until the $(k+1)$-st active order arrival can be viewed as the first success time in a series of Bernoulli experiments with success probability $\Delta p^{(n)}$ and expected value $\frac{1}{\Delta p^{(n)}}$. The $(k+1)$-st active order arrives at time

$$\sigma_{k+1}^{(n)} = \sigma_k^{(n)} + \zeta_k^{(n)} \cdot \Delta x^{(n)}$$

where

$$\zeta_k^{(n)} := \sum_{l=\sigma_k^{(n)}}^{r_{k+1}^{(n)}-1} \varphi_l \cdot \Delta p^{(n)}.$$

[12]For simplicity, it is assumed that the the initial volume density functions vanish outside a compact price interval. Hence there is a positive probability of depletion unless one assumes that no further buy/sell side price improvements take place if the distance of the current best bid/ask price from the initial state exceeds some threshold.

[13]For the results that follows we will assume that $\frac{\Delta x^{(n)} \cdot \Delta p^{(n)}}{\Delta v^{(n)}} = 1$ and $\frac{\Delta v^{(n)}}{\Delta t^{(n)}} = 1$ as $n \to \infty$. Any other constant would require further constants in the limiting dynamics.

Since the random variables $\varphi_{\sigma_k^{(n)}+1}, ..., \varphi_{r_k^{(n)}-1}$ are conditionally independent and identically distributed, $\{r_k^{(n)}\}$ and $\{\varphi_k\}$ are independent sequences. Because $\mathbb{E}[r_k^{(n)}] = \frac{1}{\Delta p^{(n)}}$, the conditional expected value $m(B^{(n)}, A^{(n)})$ of $\zeta_k^{(n)}$, given the prevailing bid and ask prices is independent of $n \in \mathbb{N}$. We assume that the mapping $m(\cdot, \cdot)$ is Lipschitz continuous.

Assumption 2.3.8. *The conditional expected value $m(B, A)$ of $\zeta_k^{(n)}$ depends in a Lipschitz continuous manner on the prevailing pair of bid and ask prices (B, A).*

We are now ready to describe the full dynamics of the state sequence. To this end, we put

$$S_k^{(n)} = \left(B_k^{(n)}, A_k^{(n)}, v_{b,k}^{(n)}, v_{s,k}^{(n)}\right)'$$

In terms of the indicator function $\mathbf{1}_k\left(S_k^{(n)}\right) := \left(\mathbf{1_A}\left(\phi_k^{(n)}\right), \dots, \mathbf{1_H}\left(\phi_k^{(n)}\right)\right)'$ the dynamics of the state sequence $\{S_k^{(n)}\}$ is of the form

$$S_{k+1}^{(n)} = S_k^{(n)} + \mathcal{D}_k^{(n)}(S_k^{(n)})$$

if we define the random operator $\mathcal{D}_k^{(n)} : E \to E$ by

$$\mathcal{D}_k^{(n)}(S_k^{(n)}) := \widetilde{\mathbb{M}}_k^{(n)}(S_k^{(n)}) \cdot \mathbf{1}_k(S_k^{(n)}) \tag{2.3.8}$$

where the matrix $\mathbb{M}_k^{(n)}(S_k^{(n)}) := \left(\widetilde{\mathbb{M}}_{\text{buy},k}^{(n)}(S_k^{(n)}), \widetilde{\mathbb{M}}_{\text{sell},k}^{(n)}(S_k^{(n)})\right)$ such that

$$\widetilde{\mathbb{M}}_{\text{buy},k}^{(n)}(S_k^{(n)}) := \begin{pmatrix} -\Delta x^{(n)} & \Delta x^{(n)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_k^{(n),A} & M_k^{(n),B} & -\frac{\Delta v^{(n)}}{\Delta x^{(n)}} M_k^{(n),C} \cdot v_{b,k}^{(n)} & \frac{\Delta v^{(n)}}{\Delta x^{(n)}} M_k^{(n),D} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\widetilde{\mathbb{M}}^{(n)}_{\mathrm{sell},k}(S^{(n)}_k) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \Delta x^{(n)} & -\Delta x^{(n)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M^{(n),E}_k & M^{(n),F}_k & -\frac{\Delta v^{(n)}}{\Delta x^{(n)}} M^{(n),G}_k \cdot v^{(n)}_{s,k} & \frac{\Delta v^{(n)}}{\Delta x^{(n)}} M^{(n),H}_k \end{pmatrix}.$$

Here, the entries referring to shifts in the volume density functions, due to best bid and ask price changes, are given by

$$\begin{aligned} M^{(n),A}_k &:= T^{(n)}_+\left(v^{(n)}_{b,k}\right) - v^{(n)}_{b,k}, \qquad & M^{(n),E}_k &:= T^{(n)}_+\left(v^{(n)}_{s,k}\right) - v^{(n)}_{s,k} \\ M^{(n),B}_k &:= T^{(n)}_-\left(v^{(n)}_{b,k}\right) - v^{(n)}_{b,k}, \qquad & M^{(n),F}_k &:= T^{(n)}_-\left(v^{(n)}_{s,k}\right) - v^{(n)}_{s,k} \end{aligned} \tag{2.3.9}$$

and the entries referring the volume changes, due to placement and cancelation of volume, are given by

$$M^{(n),I}_k(x) := \omega^I_k \sum_{j=-\infty}^{\infty} \mathbf{1}_{\{\pi^I_k \in [x^{(n)}_j, x^{(n)}_{j+1})\}}(x) \qquad \text{for events I=}\mathbf{C}, \mathbf{D}, \mathbf{G}, \mathbf{H}. \tag{2.3.10}$$

Observing the dynamics in continuous time, we define

$$S^{(n)}(t) := S^{(n)}_k \quad \text{and} \quad \tau^{(n)}(t) := \tau^{(n)}_k \quad \text{for} \quad t \in [\tau^{(n)}_k, \tau^{(n)}_{k+1}). \tag{2.3.11}$$

Remark 2.3.9. *Overall, the state and time dynamics of our models is driven by the random sequences $\{\varphi_k\}$ (event times), $\{\phi_k\}$ (event types), $\{\pi^I_k\}$ (placement/cancelation price levels) and ω^I_k (placed/canceled orders). The joint dynamics of all models can be defined in terms of suitable independent families*

$$\kappa_k := \left\{(\varphi_k(B,A), \phi_k(B,A))_{(A,B)\in\mathbb{R}^2}, (\pi^I_k, \omega^I_k)_{I=A,\dots,H}\right\} \quad (k = 0, 1, \dots)$$

of independent random variables. In particular, the process $\{(S^{(n)}(t), \tau^{(n)}(t))\}_{t\in[0,T]}$ $(n \in \mathbb{N})$ is adapted to the filtration

$$\mathcal{F}^{(n)}_k := \mathcal{F}_{\lfloor \frac{k}{\Delta t^{(n)}} \rfloor} = \sigma\left(\kappa_s : 1 \le s \le \lfloor \frac{k}{\Delta t^{(n)}} \rfloor\right).$$

2.3.2 Main Result and Proof

The main result of the section is the Weak Law of Large Numbers Theorem 2.3.10. It states that, with the choice of scaling above, the sequence of order book models indexed by n converge in probability to a continuous order book model, as $n \to \infty$. The dynamics of the limiting model can be characterized by a coupled ODE:PDE system: the dynamics of the best bid and ask prices will be given in terms of an ODE, while the relative buy and sell volume densities will be given by the respective unique classical solution of a first order linear hyperbolic PDE with variable coefficients.

Theorem 2.3.10 (Weak Law of Large Numbers for LOBs)**.** *Let $\{S^{(n)}\}_{n\geq 1}$ be the sequence of continuous time processes defined in (2.3.11) and suppose that Assumptions 2.3.2 and 2.3.3-2.3.8 hold. Then, for all $T > 0$ there exists a deterministic process $s : [0,T] \to E$ such that*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|S^{(n)}(t) - s(t)\|_E = 0 \quad \textit{in probability.}$$

The process s is of the form $s(t) = \begin{pmatrix} \gamma(t) \\ v(\cdot,t) \end{pmatrix}$, where $\gamma(t) = \begin{pmatrix} b(t) \\ a(t) \end{pmatrix}$ is the vector of the best bid and ask price at time $t \in [0,T]$ and $v(x,t) = \begin{pmatrix} v_b(x,t) \\ v_s(x,t) \end{pmatrix}$ denotes the vector of buy and sell volume densities at $t \in [0,T]$ relative to the best bid and ask price. In terms of the matrices

$$A(\cdot) := \begin{pmatrix} p^A(\cdot) - p^B(\cdot) & 0 \\ 0 & p^E(\cdot) - p^F(\cdot) \end{pmatrix}, \quad B(x,\cdot) := \begin{pmatrix} -p^C(\cdot)f^C(x) & 0 \\ 0 & -p^G(\cdot)f^G(x) \end{pmatrix}, \tag{2.3.12}$$

the vector

$$c(x,\cdot) := \begin{pmatrix} p^D(\cdot)f^D(x) \\ \\ p^H(\cdot)f^H(x) \end{pmatrix}, \tag{2.3.13}$$

and the function $m(\cdot,\cdot)$ that specifies the expected waiting time between two consecutive active order arrivals, the function γ is the unique solution to the 2-dimensional ODE system

$$\begin{cases} \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} &= \frac{A(\gamma(t))}{m(\gamma(t))}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \in [0,T] \\ \gamma(0) &= \begin{pmatrix} B_0 \\ A_0 \end{pmatrix} \end{cases} \tag{2.3.14}$$

and (v_b, v_s) *is the unique non-negative bounded classical solution of the PDE for* $(x,t) \in \mathbb{R} \times [0,T]$

$$\begin{cases} v_t(x,t) & = \frac{1}{m(\gamma(t))}\Big(A\left(\gamma(t)\right)v_x(x,t) + B(x,\gamma(t))v(x,t) + c\left(x,\gamma(t)\right)\Big), \\ v(x,0) & = v_0(x). \end{cases} \tag{2.3.15}$$

The analysis of the limiting dynamics can be simplified by separating the randomness on the level of order arrival times from that of order types as shown in the following. Subsequently, we give an explicit solution to the limiting PDE.

Proof of Theorem 2.3.10

For the continuous-time process $S^{(n)}$ we write

$$S^{(n)}(t) = \left(S_\gamma^{(n)}(t), S_v^{(n)}(t)\right)$$

where $S_\gamma^{(n)} \in \mathbb{R}^2$ describes the dynamics of bid and ask prices and $S_v^{(n)}(t) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ describes the dynamics of the buy and sell volume density functions. According to Proposition 4.2.6, the process $S^{(n)}$ can be expressed as the composition of a *state process* $\eta^{(n)}$ and a *time process* $\mu^{(n)}$:

$$S^{(n)}(t) = \eta^{(n)}\left(\mu^{(n)}(t) - \Delta t^{(n)}\right),$$

The state and time process is given by

$$\eta^{(n)}(t) := S_k^{(n)} \quad \text{for} \quad t \in \left[t_k^{(n)}, t_{k+1}^{(n)}\right) \tag{2.3.16}$$

where $t_k^{(n)} := k \cdot \Delta t^{(n)}$ and

$$y^{(n)}(u) := \tau_k^{(n)} \quad \text{for} \quad u \in \left[\tau_k^{(n)}, \tau_{k+1}^{(n)}\right), \tag{2.3.17}$$

respectively. The time-change $\mu^{(n)}$ is then defined in terms of $y^{(n)}$ as

$$\mu^{(n)}(t) := \inf\{u > 0 : y^{(n)}(u) > t\}.$$

The advantage of the state and time separation is that the processes $\eta^{(n)}$ and $\mu^{(n)}$ can be analyzed separately. In fact, we will show convergence in probability

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\|\eta^{(n)}(t) - \eta(t)\right\|_E = 0 \quad \text{and} \quad \lim_{n\to\infty} \sup_{t\in[0,T]} \left|\mu^{(n)}(t) - \mu(t)\right| = 0$$

to limiting processes $\eta(t) = (\eta_\gamma(t), \eta_v(t))$ and $\mu(t)$. Since the state sequence takes values in the Hilbert space E, the Time Change Theorem 4.2.7 implies that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|S^{(n)}(t) - \eta(\mu(t))\|_E = 0 \quad \text{in probability.}$$

In our model, bid and ask prices are a sufficient statistic for the evolution of the order book. In particular, the limiting behavior of the sequences $\eta_\gamma^{(n)}$ and $\mu^{(n)}$ can be analyzed without reference to volumes. We will prove the following proposition.

Proposition 2.3.11. *Let $\hat{\gamma}$ be the unique solution to the ODE*

$$\begin{cases} \frac{\mathrm{d}\hat{\gamma}(t)}{\mathrm{d}t} = A(\hat{\gamma}(t)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & t \in (0,T] \\ \\ \hat{\gamma}(0) = \begin{pmatrix} B_0 \\ A_0 \end{pmatrix}. \end{cases} \tag{2.3.18}$$

Then

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left|\eta_\gamma^{(n)}(t) - \hat{\gamma}(t)\right| = 0 \quad \textit{in probability.}$$

Moreover, the sequence of processes $\mu^{(n)}$ satisfies

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left|\mu^{(n)}(t) - \mu(t)\right| = 0 \quad \textit{in probability.} \qquad \textit{where} \qquad \mu^{-1}(t) = \int_0^t m(\hat{\gamma}_u)du.$$

Once the limiting time-change process μ has been identified, what remains to finish the proof of Theorem 2.3.10, is to establish convergence of the volume processes $\eta_v^{(n)}$ to their deterministic limit. This will be achieved below, where we prove the following result.

Proposition 2.3.12. *Let $\widehat{u}$ be the unique classical solution of the PDE system*

$$\begin{cases} \widehat{u}_t(x,t) & = A\left(\widehat{\gamma}(t)\right)\widehat{u}_x(x,t) + B(x,\widehat{\gamma}(t))\widehat{u}(x,t) + c\left(x,\widehat{\gamma}(t)\right), & (x,t) \in \mathbb{R}\times[0,T] \\ \widehat{u}(x,0) & = v_0(x), & x \in \mathbb{R} \end{cases}. \tag{2.3.19}$$

Then

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left\|\eta_v^{(n)}(\cdot,t) - \widehat{u}(\cdot,t)\right\|_{L^2} = 0 \quad \textit{in probability.}$$

Explicit solution of the PDE system.

The explicit solution of the PDE system (2.3.19) may be found by the method of characteristics that was employed in the proof of Proposition 2.2.6. Thus, in particular, the equations for the buy and sell side can be solved independently and the representation is given by equations analogous to (2.2.47)-(2.2.49).

Due to our smoothness assumptions on the volume placement and cancelation functions it is not hard to verify that the solution is uniformly bounded with uniformly bounded first and second derivatives with respect to the time and space variable. Moreover, since the function $v_{b,0}$ vanishes outside a compact interval (Assumption 2.3.2) and no orders are placed or canceled beyond a distance M form the best bid/ask price (Assumption 2.3.3), the function $u_b(t, \cdot)$ vanishes outside some compact interval $I(T)$ for all $t \in [0, T]$. Altogether, one has the following result.

Proposition 2.3.13. *Under the assumptions of Theorem 2.3.10, the PDE (2.3.15) has a unique solution v. The solution is uniformly bounded, with uniformly bounded first and second derivatives with respect to both variables and there exists an interval I such that $v(x,t) = \mathbf{0}$ for all $t \in [0,T]$ and $x \notin I$.*

Convergence of bid/ask prices.

According to Proposition 4.2.6, the process $S^{(n)}$ can be represented in terms of a composition of a state process $\eta^{(n)}$ that jumps at deterministic times $\{t_k^{(n)}\}$ and a time process $\mu^{(n)}$ that accounts for the random event arrival times. Prices change less frequently at times $\{\sigma_k^{(n)}\}$. This suggests to introduce a second time scale, which will be referred to as *active order time*, defined by

$$s_k^{(n)} := k \cdot \Delta x^{(n)}$$

along which to scale the price process. In order to make this more precise, let us denote by $\mathcal{D}_{\gamma,k}^{(n)}$ the restriction of the operator $\mathcal{D}_k^{(n)}$ to the price component of the state sequence and put

$$\widehat{\mathcal{D}}_{\gamma,k}^{(n)} := \sum_{l=\sigma_{k-1}^{(n)}}^{\sigma_k^{(n)}-1} \mathcal{D}_{\gamma,l}^{(n)} = \mathcal{D}_{\gamma,\sigma_k^{(n)}}^{(n)}$$

where the second equality follows from the fact that prices do not change between times $\sigma_k^{(n)}$ and $\sigma_{k+1}^{(n)} - 1$. Furthermore, we introduce the family of continuous time stochastic processes $\widehat{\eta}_\gamma^{(n)}$ defined by

$$\widehat{\eta}^{(n)}(t) := \widehat{\eta}_k^{(n)} \quad \text{for } t \in [s_k^{(n)}, s_{k+1}^{(n)})$$

where

$$\begin{cases} \widehat{\eta}^{(n)}_{\gamma,k+1} & := \widehat{\eta}^{(n)}_{\gamma,k} + \widehat{\mathcal{D}}^{(n)}_{\gamma,k}(\widehat{\eta}^{(n)}_k) \\ \\ \widehat{\eta}^{(n)}_0 & = \begin{pmatrix} B^{(n)}_0 \\ A^{(n)}_0 \end{pmatrix}. \end{cases} \tag{2.3.20}$$

The quantity $\widehat{\eta}^{(n)}_{\gamma,k}$ describes the state of the price process after the k:th price change. The following lemma shows that the process $\eta^{(n)}_\gamma$, evolving on the level of event time, and the process $\widehat{\eta}^{(n)}_\gamma$, evolving on the level of active order time, are indistinguishable in the limit when $n \to \infty$.

Lemma 2.3.14. *For any $T > 0$ and $\epsilon > 0$, it holds that*

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\le t\le T} \left| \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \widehat{\mathcal{D}}^{(n)}_{\gamma,k} - \sum_{k=0}^{\lfloor \frac{t}{\Delta t^{(n)}} \rfloor} \mathcal{D}^{(n)}_{\gamma,k} \right| > \epsilon \right) = 0.$$

Proof. By construction, the two sums $\sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \widehat{\mathcal{D}}^{(n)}_{\gamma,k}$ and $\sum_{k=0}^{\lfloor \frac{t}{\Delta t^{(n)}} \rfloor} \mathcal{D}^{(n)}_{\gamma,k}$ have the same expected value for any $t \in [0, T]$:

$$\sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \mathbb{E}\widehat{\mathcal{D}}^{(n)}_{\gamma,k} = \sum_{k=0}^{\lfloor \frac{t}{\Delta t^{(n)}} \rfloor} \mathbb{E}\mathcal{D}^{(n)}_{\gamma,k}.$$

As a result, it is enough to prove that

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\le t\le T} \left| \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \left\{ \widehat{\mathcal{D}}^{(n)}_{\gamma,k} - \mathbb{E}\widehat{\mathcal{D}}^{(n)}_{\gamma,k} \right\} \right| > \frac{\epsilon}{2} \right) \\ &= \lim_{n\to\infty} \mathbb{P}\left(\sup_{0\le t\le T} \left| \sum_{k=0}^{\lfloor \frac{t}{\Delta t^{(n)}} \rfloor} \left\{ \mathcal{D}^{(n)}_{\gamma,k} - \mathbb{E}\mathcal{D}^{(n)}_{\gamma,k} \right\} \right| > \frac{\epsilon}{2} \right) = 0. \end{aligned}$$

The random variables

$$\frac{1}{\Delta x^{(n)}} \left\{ \widehat{\mathcal{D}}^{(n)}_{\gamma,k} - \mathbb{E}\widehat{\mathcal{D}}^{(n)}_{\gamma,k} \right\}, \quad k = 0, ..., \lfloor \frac{T}{\Delta x^{(n)}} \rfloor, \quad n \in \mathbb{N}$$

and

$$\frac{1}{\Delta t^{(n)}} \left\{ \mathcal{D}^{(n)}_{\gamma,k} - \mathbb{E}\mathcal{D}^{(n)}_{\gamma,k} \right\}, \quad k = 0, ..., \lfloor \frac{T}{\Delta t^{(n)}} \rfloor, \quad n \in \mathbb{N}$$

form triangular martingale difference arrays in the sense of Definition 5.2.6 in the Appendix with respect to the filtrations $\{\mathcal{F}_{\sigma_k^{(n)}}\}_{k\in\mathbb{N}}$ and $\{\mathcal{F}_{k\in\mathbb{N}}\}_k$, respectively. A direct computation shows that they are uniformly L^2-bounded. Thus, it follows from Theorem 5.2.8 in the Appendix that for all $\alpha > \frac{1}{2}$:

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\le m\le \lfloor \frac{T}{\Delta x^{(n)}} \rfloor} \frac{1}{\Delta x^{(n)}} \left| \sum_{k=0}^{m} \left\{ \widehat{\mathcal{D}}_{\gamma,k}^{(n)} - \mathbb{E}\widehat{\mathcal{D}}_{\gamma,k}^{(n)} \right\} \right| \ge \frac{\epsilon}{2} \left(\frac{T}{\Delta x^{(n)}} \right)^{\alpha} \right) = 0$$

as well as

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\le m\le \lfloor \frac{T}{\Delta t^{(n)}} \rfloor} \frac{1}{\Delta t^{(n)}} \left| \sum_{k=0}^{m} \left\{ \mathcal{D}_{\gamma,k}^{(n)} - \mathbb{E}\mathcal{D}_{\gamma,k}^{(n)} \right\} \right| \ge \frac{\epsilon}{2} \left(\frac{T}{\Delta t^{(n)}} \right)^{\alpha} \right) = 0.$$

Choosing $\alpha \in (\frac{1}{2}, 1)$ and multiplying the inequalities in the above probabilities by $\Delta x^{(n)}$ and $\Delta t^{(n)}$, respectively, proves the assertion. $\square$

Let $\widehat{\gamma}$ be the solution to the ODE (2.3.18) and consider the discretisation $\widehat{\gamma}_k^{(n)} := \widehat{\gamma}(s_k^{(n)})$. The next lemma shows that the sequence of expected price processes $\widetilde{\gamma}^{(n)}$ defined by

$$\widetilde{\gamma}^{(n)}(t) := \widetilde{\gamma}_k^{(n)} \quad \text{for } t \in [s_k^{(n)}, s_{k+1}^{(n)})$$

where

$$\begin{cases} \widetilde{\gamma}_{k+1}^{(n)} & := \widetilde{\gamma}_k^{(n)} + \mathbb{E}\left[\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widetilde{\gamma}_k^{(n)}) \right] \\ \\ \widetilde{\gamma}_0^{(n)} & = \begin{pmatrix} B_0^{(n)} \\ A_0^{(n)} \end{pmatrix} \end{cases} \tag{2.3.21}$$

converges uniformly to $\widehat{\gamma}$ on compact time intervals.

Lemma 2.3.15. *For any $T > 0$*

$$\sup_{t\in[0,T]} |\widetilde{\gamma}^{(n)}(t) - \widehat{\gamma}(t)| = \mathcal{O}(\Delta x^{(n)}) \quad a.s.\ as\ n \to \infty.$$

Proof. The proof is completely analogous to that of Lemma 2.2.1. $\square$

We are now ready to prove convergence of bid and ask prices in probability.

Proof of Proposition 2.3.11.

a) We first consider the convergence of the state process $\eta_\gamma^{(n)}$ and claim that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\eta_\gamma^{(n)}(t) - \widehat{\gamma}(t)| \to 0, \quad \text{in probability.} \tag{2.3.22}$$

In view of Lemma 2.3.14, we can write

$$\begin{aligned}\eta_\gamma^{(n)}(t) &= \widehat{\eta}_\gamma^{(n)}(t) \\ &= \gamma^{(n)}(0) + \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \widehat{\mathcal{D}}_{\gamma,k}^{(n)}\left(\widehat{\eta}_{\gamma,k}^{(n)}\right) \\ &= \gamma^{(n)}(0) + \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \mathbb{E}\left[\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\eta}_{\gamma,k}^{(n)})\right] + \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \left(\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\eta}_{\gamma,k}^{(n)}) - \mathbb{E}\left[\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\eta}_{\gamma,k}^{(n)})\right]\right)\end{aligned}$$

up to some random additive constant that vanishes almost surely uniformly in $t \in [0,T]$ as $n \to \infty$. Adding and subtracting the sequence $\widetilde{\gamma}^{(n)}$ yields (again up to a vanishing additive constant):

$$\begin{aligned}\left|\eta_\gamma^{(n)}(t) - \widehat{\gamma}(t)\right| \;\leq\; & \left|\widetilde{\gamma}^{(n)}(t) - \widehat{\gamma}(t)\right| \\ & + \left|\sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \mathbb{E}\left[\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\eta}_{\gamma,k}^{(n)})\right] - \widetilde{\gamma}^{(n)}(t)\right| \\ & + \left|\sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \left(\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\eta}_{\gamma,k}^{(n)}) - \mathbb{E}\left[\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\eta}_{\gamma,k}^{(n)})\right]\right)\right|.\end{aligned}$$

For the first term, we deduce from Lemma 2.3.15 that

$$\sup_{t\in[0,T]} |\widetilde{\gamma}^{(n)}(t) - \widehat{\gamma}(t)| = \mathcal{O}(\Delta t^{(n)}). \tag{2.3.23}$$

For the second term we use the Lipschitz continuity of the event probabilities $p^I(\cdot,\cdot)$ in order to establish the existence of a constant $L_\gamma > 0$ such that:

$$\begin{aligned}\left|\sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \mathbb{E}\left[\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widetilde{\eta}_{\gamma,k}^{(n)})\right] - \widetilde{\gamma}^{(n)}(t)\right| &= \left|\sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \mathbb{E}\left[\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\eta}_{\gamma,k}^{(n)})\right] - \mathbb{E}\left[\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\gamma}_k^{(n)})\right]\right| \\ &\leq \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \left|\mathbb{E}\left[\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\eta}_{\gamma,k}^{(n)})\right] - \mathbb{E}\left[\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\gamma}_k^{(n)})\right]\right| \\ &\leq \Delta x^{(n)} \cdot L_\gamma \cdot \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \left|\widehat{\eta}_{\gamma,k}^{(n)} - \widehat{\gamma}_k^{(n)}\right|.\end{aligned}$$

The third term corresponds to the noise-term of the price process. For each $n \in \mathbb{N}$, the sequence

$$y_k^n := \widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\eta}_{\gamma,k}^{(n)}) - \mathbb{E}\left[\widehat{\mathcal{D}}_{\gamma,k}^{(n)}(\widehat{\eta}_{\gamma,k}^{(n)})\right], \quad k = 0, ..., \lfloor \frac{T}{\Delta x^{(n)}} \rfloor$$

is a martingale difference sequence. A direct computation shows that

$$\sup_{n,k} \mathbb{E}|y_k^n|^2 \leq C \cdot \left(\Delta x^{(n)}\right)^2.$$

Hence, the law of large numbers for triangular martingale difference arrays (Theorem 5.2.8 and Corollary 5.2.9 in the Appendix) implies

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0 \leq m \leq \frac{T}{\Delta x^{(n)}}} |\sum_{k=0}^{m} y_k^n| > 0 \right) = 0,$$

just as in the proof of Lemma 2.3.14. Thus, using Lemma 2.3.14 again, we see that

$$\begin{aligned} \left|\eta_\gamma^{(n)}(t) - \widehat{\gamma}(t)\right| &= \left|\widehat{\eta}_\gamma^{(n)}(t) - \widehat{\gamma}(t)\right| \\ &\leq \Delta x^{(n)} \cdot L_\gamma \cdot \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \left|\widehat{\eta}_{\gamma,k}^{(n)} - \widehat{\gamma}_k^{(n)}\right| + o(1) \quad \text{in probability} \end{aligned}$$

for some additive term of order $o(1)$ uniform in $t \in [0, T]$. As a result, (2.3.22) follows from an application of Gronwall's lemma along with Lemma 2.3.14.

b) Let us now consider the cumulative "active order time process"

$$\begin{aligned} y^{(n)}(t) &:= \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \zeta^{(n)}\left(\widehat{\eta}_{\gamma,k}^{(n)}\right) \cdot \Delta x^{(n)} \\ &= \Delta x^{(n)} \cdot \left\{ \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \mathbb{E}\left[\zeta^{(n)}\left(\widehat{\eta}_{\gamma,k}^{(n)}\right)\right] + \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \left(\zeta^{(n)}\left(\widehat{\eta}_{\gamma,k}^{(n)}\right) - \mathbb{E}\left[\zeta^{(n)}\left(\widehat{\eta}_{\gamma,k}^{(n)}\right)\right]\right) \right\} \\ &= \Delta x^{(n)} \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} m\left(\widehat{\eta}_{\gamma,k}^{(n)}\right) + \Delta x^{(n)} \cdot \sum_{k=0}^{\lfloor \frac{t}{\Delta x^{(n)}} \rfloor} \left(\zeta^{(n)}\left(\widehat{\eta}_{\gamma,k}^{(n)}\right) - \mathbb{E}\left[\zeta^{(n)}\left(\widehat{\eta}_{\gamma,k}^{(n)}\right)\right]\right). \end{aligned} \tag{2.3.24}$$

By the above established uniform convergence of $\widehat{\eta}_\gamma^{(n)}$ to $\widehat{\gamma}$ in probability and because the function m is Lipschitz continuous, the first sum converges to the

function
$$y(t) = \int_0^t m(\widehat{\gamma}(u))\mathrm{d}u. \tag{2.3.25}$$
Applying the same arguments as above to the martingale difference sequences
$$\zeta^{(n)}\left(\widehat{\eta}_{\gamma,k}^{(n)}\right) - \mathbb{E}\left[\zeta_k^{(n)}\left(\widehat{\eta}_{\gamma,k}^{(n)}\right)\right], \quad k = 0, ..., \lfloor\frac{t}{\Delta x^{(n)}}\rfloor$$
we see that the second term vanishes uniformly in $t \in [0, T]$ in probability. Thus,
$$\lim_{n\to\infty} \sup_{t\in[0,T]} |y^{(n)}(t) - y(t)| = 0 \quad \text{in probability.}$$

Since $y^{(n)}$ and y are increasing functions, their inverses $\mu^{(n)}$ and μ exist. By continuity
$$\sup_{t\in[0,T]} |\mu^{(n)}(t) - \mu(t)| \to 0 \quad \text{in probability as } n \to \infty$$
and
$$\mu'(t) = \left(y^{-1}(t)\right)' = \frac{1}{y'(y^{-1}(t))} = \frac{1}{m(\mu(t))} = \frac{1}{m(\widehat{\gamma}(\mu(t)))}. \tag{2.3.26}$$

Since both the state and the time process converge, we conclude from the time change theorem that
$$\sup_{t\in[0,T]} \left|\eta_\gamma^{(n)}(t) - \gamma(t)\right| \to 0 \quad \text{in probability as } n \to \infty,$$
where $\gamma(t) = \widehat{\gamma}(\mu(t))$ and
$$\gamma'(t) = \widehat{\gamma}'(\mu(t)) \cdot \mu'(t) = \frac{A(\widehat{\gamma}(\mu(t)))}{m(\widehat{\gamma}(\mu(t)))}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{A(\gamma(t))}{m(\gamma(t))}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

□

Convergence of volume densities.

In this part of we prove Proposition 2.3.12. To this end, we denote by $\mathcal{D}_{v,k}^{(n)}(\cdot,\cdot)$ the restriction of the operator $\mathcal{D}_k^{(n)}$ to $L^2 \times L^2$, i.e the restriction of $\mathcal{D}_k^{(n)}$ to the volume components of the state process. We need to show that the sequence $\{\eta_v^{(n)}\}_{n\in\mathbb{N}}$ of $L^2 \times L^2$-valued step-functions defined recursively by
$$\eta_v^{(n)}(t, \cdot) := \eta_{v,k}^{(n)} \quad \text{for } t \in [t_k^{(n)}, t_{k+1}^{(n)}) \tag{2.3.27}$$

where

$$\begin{cases} \eta_{v,k+1}^{(n)} & := \eta_k^{(n)} + \mathcal{D}_{v,k}^{(n)} \left(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)} \right) \\ \\ \eta_{v,0}^{(n)} & := v_0^{(n)} \end{cases} \tag{2.3.28}$$

converges in probability in L^2 to the unique solution of the PDE (2.3.19). We will show convergence in several steps. In a first step we find a convergent discretization scheme of the PDE which is coherent with the order book dynamics. Subsequently, we link this scheme to the expected dynamics of the volume densities.

A numerical scheme for the limiting PDE.

For any $n \in \mathbb{N}$, the scaling parameters $\Delta x^{(n)}$ and $\Delta t^{(n)}$ define a grid $\{(t_k^{(n)}, x_k^{(n)})\}$ on $[0,T] \times \mathbb{R}$ through $t_k^{(n)} = k \cdot \Delta t^{(n)}$ ($k \in \mathbb{N}_0$) and $x_j^{(n)} = j \cdot \Delta x^{(n)}$ ($j \in \mathbb{Z}$). In a first step, we approximate the unique solution $\widehat{u} : [0,T] \times \mathbb{R} \to \mathbb{R}^2$ to (2.3.19) through a sequence of grid-point functions $\widehat{u}^{(n)} : [0,T] \times \mathbb{R} \to \mathbb{R}^2$. To this end, we put

$$A_R(t) := \begin{pmatrix} p^A(\hat{\gamma}(t)) & 0 \\ 0 & p^E(\hat{\gamma}(t)) \end{pmatrix}, \quad A_L(t) := \begin{pmatrix} p^B(\hat{\gamma}(t)) & 0 \\ 0 & p^F(\hat{\gamma}(t)) \end{pmatrix}$$

and

$$F(t,x) := B(\hat{\gamma}(t), x), \quad g(t,x) := c(\hat{\gamma}(t), x).$$

Furthermore, we introduce operators $\mathcal{H}_t^{(n)}$ that act on $v \in L^2$ according to

$$\begin{aligned} \mathcal{H}_t^{(n)}(v) := v &+ \Delta p^{(n)} \cdot A_R(t) \left[v(\cdot + \Delta x^{(n)}) - v(\cdot) \right] \\ &+ \Delta p^{(n)} \cdot A_L(t) \left[v(\cdot - \Delta x^{(n)}) - v(\cdot) \right] \\ &+ \Delta v^{(n)} \cdot (1 - \Delta p^{(n)}) \cdot \left[F(t, \cdot) \cdot v(\cdot) + g(t, \cdot) \right]. \end{aligned}$$

The sequence of grid-point approximations is then defined recursively by

$$\widehat{u}^{(n)}(\cdot, t) := \widehat{u}_k^{(n)} \quad \text{for} \quad t \in [t_k^{(n}, t_{k+1}^{(n)}) \tag{2.3.29}$$

where

$$\begin{cases} \widehat{u}_{k+1}^{(n)} & = \mathcal{H}_{t_k^{(n)}}^{(n)}(\widehat{u}_k^{(n)}) \\ \\ \widehat{u}_0^{(n)} & = v_0^{(n)}. \end{cases} \tag{2.3.30}$$

The sequence of step-functions $\{\widehat{u}^{(n)}\}$ essentially describes a discretized limiting volume dynamics of the order book. We benchmark this dynamics against the expected pre-limit volume dynamics when prices are replaced by the limiting dynamics. More precisely, we introduce another sequence of step functions $u^{(n)} : [0,T] \times \mathbb{R} \to \mathbb{R}$ by

$$u^{(n)}(\cdot,t) := u_k^{(n)} \quad \text{for} \quad t \in [t_k^{(n}, t_{k+1}^{(n)}) \tag{2.3.31}$$

where

$$\begin{cases} u_{k+1}^{(n)} & = u_k^{(n)} + \mathbb{E}\left[\mathcal{D}_{v,k}^{(n)}\left(\hat{\gamma}(t_k^{(n)}), u_k^{(n)}\right)\right] \\ \\ u_0^{(n)} & := v_0^{(n)}. \end{cases} \tag{2.3.32}$$

In a first step we are now going to show that the grid-point functions $\widehat{u}^{(n)}$ approximate the solution $\widehat{u}$ of our PDE. Subsequently we show that the PDE can as well be approximated by the functions $u^{(n)}$.

Proposition 2.3.16 (Convergence of the numerical scheme)**.** *Assume that the assumptions of Theorem 2.3.10 hold. Then, the processes $\widehat{u}^{(n)}$ define a convergent finite difference scheme of the PDE (2.3.19), i.e.*

$$\sup_{t\in[0,T]} \|\widehat{u}^{(n)}(t,\cdot) - \widehat{u}(t,\cdot)\|_{L^2} \to 0 \quad \textit{as } n \to \infty. \tag{2.3.33}$$

Proof. The proof is analogous to that of Proposition 2.2.6.

□

Next, we show that the functions $u^{(n)}$ also approximate the PDE. More precisely, the following holds.

Proposition 2.3.17. *Under the assumptions of Theorem 2.3.10*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\widehat{u}^{(n)}(t;\cdot) - u^{(n)}(t;\cdot)\|_{L^2} = 0.$$

Proof. The proof is similar to those of Propositions 2.2.6 and 2.3.16. By analogy to the operator $\mathcal{H}_t^{(n)}$ we introduce an operator $\widehat{\mathcal{H}}_t^{(n)}$ on L^2 by

$$\begin{aligned} \widehat{\mathcal{H}}_t^{(n)}(v) := v &+ \Delta p^{(n)} \cdot A_R(t)\left[v(\cdot + \Delta x^{(n)}) - v(\cdot)\right] \\ &+ \Delta p^{(n)} \cdot A_L(t)\left[v(\cdot - \Delta x^{(n)}) - v(\cdot)\right] \\ &+ \Delta v^{(n)} \cdot (1 - \Delta p^{(n)})\left[F^{(n)}(t,\cdot)\cdot v(\cdot) + g^{(n)}(t,\cdot)\right]. \end{aligned}$$

where for $t \in [t_k^{(n)}, t_{k+1}^{(n)})$:

$$\begin{aligned}
F^{(n)}(t,\cdot) &:= \begin{pmatrix} -f^{(n),C}(\cdot)\cdot p^C(\widehat{\gamma}(t_k^{(n)})) & 0 \\ 0 & -f^{(n),G}(\cdot)\cdot p^G(\widehat{\gamma}(t_k^{(n)})) \end{pmatrix} \\
g^{(n)}(t,\cdot) &:= \begin{pmatrix} f^{(n),D}(\cdot)\cdot p^D(\widehat{\gamma}(t_k^{(n)})) \\ f^{(n),H}(\cdot)\cdot p^H(\widehat{\gamma}(t_k^{(n)})) \end{pmatrix}.
\end{aligned}$$

For the error function $\delta\widehat{u}_k^{(n)}(\cdot) := \widehat{u}_k^{(n)}(\cdot) - u_k^{(n)}(\cdot)$ we then obtain

$$\begin{aligned}
\delta\widehat{u}_{k+1}^{(n)}(\cdot) &= \widehat{\mathcal{H}}_t^{(n)}(\delta\widehat{v}_k^{(n)}(\cdot)) + \Delta v^{(n)}\cdot(1-\Delta p^{(n)})\cdot\left(\delta F^{(n)}(\cdot)\cdot\widehat{u}_k^{(n)}(\cdot) + \delta g^{(n)}(\cdot)\right) \\
&=: \widehat{\mathcal{H}}_t^{(n)}(\delta\widehat{u}_k^{(n)}(\cdot)) + \Delta v^{(n)}\cdot(1-\Delta p^{(n)})\cdot\widehat{\mathcal{L}}(t_k^{(n)};\cdot)
\end{aligned}$$

where

$$\begin{aligned}
\delta F^{(n)}(t,\cdot) &:= \begin{pmatrix} -(f^{(n),C}(\cdot)-f^C(\cdot))\cdot p^C(\widehat{\gamma}(t_k^{(n)})) & 0 \\ 0 & -(f^{(n),G}(\cdot)-f^G(\cdot))\cdot p^G(\widehat{\gamma}(t_k^{(n)})) \end{pmatrix}, \\
\delta g^{(n)}(t,\cdot) &:= \begin{pmatrix} \left(f^{(n),D}(\cdot)-f^D(\cdot)\right)\cdot p^D(\widehat{\gamma}(t_k^{(n)})) \\ \left(f^{(n),H}(\cdot)-f^H(\cdot)\right)\cdot p^H(\widehat{\gamma}(t_k^{(n)})) \end{pmatrix}.
\end{aligned}$$

By construction, the grid-point functions $\widehat{u}^{(n)}$ are uniformly bounded. As it result, it follows from Assumption 2.3.3 that

$$\lim_{n\to\infty}\sup_{k=0,\dots,\lfloor\frac{T}{\Delta t^{(n)}}\rfloor}\|\widehat{\mathcal{L}}(t_k^{(n)};\cdot)\|_{L^2} = 0.$$

One can now proceed as in the proof of Proposition 2.3.16, to conclude that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|\widehat{u}^{(n)}(t,\cdot) - u^{(n)}(t;\cdot)\|_{L^2} = 0.$$

□

Expected volume dynamics and discretized PDEs.

To show the convergence of volume density functions we compare the random states $\eta_v^{(n)}$ with the deterministic approximations of the limiting PDE obtained in the previous subsection. For this, we introduce the deterministic step function valued processes $\widetilde{u}^{(n)}$:

$$\widetilde{u}^{(n)}(\cdot,t) := \widetilde{u}_k^{(n)} \quad \text{for } t\in[t_k^{(n}, t_{k+1}^{(n)}) \tag{2.3.34}$$

where

$$\begin{cases} \widetilde{u}_{k+1}^{(n)} & := \widetilde{u}_k^{(n)} + \mathbb{E}\left[\mathcal{D}_{v,k}^{(n)}\left(\eta_{\gamma,k}^{(n)}, \widetilde{u}_k^{(n)}\right)\right] \\ \widetilde{u}_0^{(n)} & := v_0^{(n)}. \end{cases} \tag{2.3.35}$$

The process $\widetilde{u}^{(n)}$ describes the expected dynamics of the volume density functions for the actual price process; in particular, $\widetilde{u}^{(n)}$ is a stochastic process. By contrast, the process $u^{(n)}$ describes the dynamics of the expected volume density functions when the random evolution of bid and ask prices is replaced by its deterministic limit. We have:

$$\begin{aligned} \|\eta_v^{(n)}(\cdot,t) - \widehat{u}(\cdot,t)\|_{L^2} \quad \leq \quad & \|\eta_v^{(n)}(\cdot,t) - \widetilde{u}^{(n)}(\cdot,t)\|_{L^2} + \|\widetilde{u}^{(n)}(\cdot,t) - u^{(n)}(\cdot,t)\|_{L^2} \\ & + \|u^{(n)}(\cdot,t) - \widehat{u}^{(n)}(\cdot,t)\| + \|\widehat{u}^{(n)}(\cdot,t) - \widehat{u}(\cdot,t)\|_{L^2}. \end{aligned}$$

The last two terms are deterministic and converge uniformly to zero by Propositions 2.3.16 and 2.3.17. It remains to show convergence of the first two (random) terms. This will be achieved in the following two subsections.

Estimating the price impact of expected volume dynamics.

The term $\|\widetilde{u}^{(n)}(\cdot,t) - u^{(n)}(\cdot,t)\|_{L^2}$ measures the impact of the noise in the price process on the expected standing volume. The following proposition shows that it converges to zero almost surely (i.e. for almost all price processes), uniformly over compact time intervals.

Proposition 2.3.18. *Under the assumptions of Theorem 2.3.10 it holds that*

$$\sup_{t\in[0,T]} \|\widetilde{u}^{(n)}(\cdot,t) - u^{(n)}(\cdot,t)\|_{L^2} \to 0 \quad a.s. \ as \ n \to \infty.$$

Proof. Let us introduce the error function

$$\delta u_k^{(n)} := u_k^{(n)} - \widetilde{u}_k^{(n)},$$

as well as

$$\delta p_k^{(n),I} := p^I(\hat{\gamma}(t_k^{(n)})) - p^I(\eta_{\gamma,k}^{(n)}) \quad \text{for} \quad I = \mathbf{A}, \mathbf{B}, \mathbf{E}, \mathbf{F}.$$

We put

$$\delta A_{R,k}^{(n)} := \begin{pmatrix} \delta p_k^{(n),A} & 0 \\ 0 & \delta p_k^{(n),E} \end{pmatrix}, \quad \delta A_{L.k}^{(n)} := \begin{pmatrix} \delta p_k^{(n),B} & 0 \\ 0 & \delta p_k^{(n),F} \end{pmatrix}$$

and denote by $\delta B_k^{(n)}$ and $\delta c_k^{(n)}$ the corresponding quantities for cancelations and limit order placements. Then, $\delta u_0^{(n)} = 0$ and

$$\begin{aligned}\delta u_{k+1}^{(n)} &= \Delta p^{(n)} \cdot \delta A_{R,k}^{(n)} \cdot (T_+^{(n)} u_k^{(n)} - u_k^{(n)}) + \Delta p^{(n)} \cdot A_{R,k}^{(n)} \cdot (T_+^{(n)}(\delta u_k^n) - \delta u_k^n) \\ &+ \Delta p^{(n)} \cdot \delta A_{L,k}^{(n)} \cdot (T_-^{(n)} u_k^{(n)} - u_k^{(n)}) + \Delta p^{(n)} \cdot A_{L,k}^{(n)} \cdot (T_-^{(n)}(\delta u_k^n) - \delta u_k^n) \\ &+ (1 - \Delta p^{(n)}) \left[\Delta v^{(n)} \cdot \delta B_k^{(n)} u_k^{(n)} + \Delta v^{(n)} \cdot B_k^{(n)} \delta u_k^{(n)} + \Delta v^{(n)} \cdot \delta c_k^{(n)} \right].\end{aligned}$$

Corollary 5.2.12 establishes

$$\|u_k^{(n)}\|_{L^2} \leq L \quad \text{and} \quad \|T_\pm^{(n)} u_k^{(n)} - u_k^{(n)}\|_{L^2} \leq L \cdot \Delta x^{(n)}$$

for some constant $L < \infty$ that is independent of (n, k). Using our boundedness assumptions on the placements and cancelations functions $f^{(n),C,D,G,H}$ along with the fact that $\Delta p^{(n)} \cdot \Delta x^{(n)} = \Delta v^{(n)} = \Delta t^{(n)}$ this shows that the L^2-norms of the first, third, fifth and seventh term on the right-hand side of the above equation are bounded by a term of the form $C \Delta t^{(n)}$ for some constant $C > 0$ that is independent of (n, k).

The sixth term can be estimated from above by a term of the form $C \Delta t^{(n)} \|\delta u^{(n)}\|_{L^2}$. Using the isometry property of the translation operator, the norm of the second and fourth term can be estimated from above by a term of the form $C \Delta p^{(n)} \|\delta u^{(n)}\|_{L^2}$. Altogether, there are constants $C_1 > 0$ and $C_2 > 0$ that do neither depend on $n \in \mathbb{N}$, nor on $k = 0, ..., \lfloor T/\Delta t^{(n)} \rfloor$, such that

$$\|\delta u_{k+1}^{(n)}\|_{L^2} \leq C_1 \Delta t^{(n)} + C_2 \Delta p^{(n)} \|\delta u_k^{(n)}\|_{L^2} = C_1 \Delta p^{(n)} \left(\frac{\Delta t^{(n)}}{\Delta p^{(n)}} + \frac{C_2}{C_1} \|\delta u_k^{(n)}\|_{L^2} \right) \quad \text{a.s.}$$

We may with no loss of generality assume that $\alpha := \frac{C_2}{C_1} < 1$. Let us then consider the deterministic difference equation

$$y_{k+1} = 1 + \alpha \cdot y_k, \quad y_0 = 0.$$

The sequence $\{y_k\}$ converges to $\frac{1}{1-\alpha}$ as $k \to \infty$. In particular $\sup_k y_k < \infty$. By construction $\|\delta u_0^{(n)}\|_{L^2} = 0$. Thus, a straightforward induction argument shows that $\|\delta u_k^{(n)}\|_{L^2} \leq C_1 \cdot \Delta p^{(n)} \cdot y_k$ almost surely. Hence

$$\lim_{n \to \infty} \sup_{k=0,\dots,\lfloor \frac{T}{\Delta t^{(n)}} \rfloor} \|\delta u_k^{(n)}\|_{L^2} \leq \lim_{n \to \infty} C_1 \cdot \Delta p^{(n)} \cdot \sup_k y_k = 0 \quad \text{a.s.}$$

□

Convergence of volumes to their expected values.

In this part, we apply a law of large number for Hilbert space-valued triangular martingale difference arrays (TMDAs) in order to establish the missing convergence to zero of the distance between $\eta_v^{(n)}$ and $\widetilde{u}^{(n)}$. More precisely, our goal is to prove the following result.

Proposition 2.3.19. *Suppose the assumptions of Theorem 2.3.10 hold. Then,*

$$\sup_{t\in[0,T]} \|\eta_v^{(n)}(\cdot,t) - \widetilde{u}^{(n)}(t,\cdot)\|_{L^2} \to 0 \quad \textit{in probability} \quad \textit{as } n\to\infty.$$

Proof. Using the definition of $\eta_{v,k}^{(n)}$ in (2.3.28) and $\widetilde{u}_k^{(n)}$ in (2.3.34) we see that

$$\widetilde{u}_k^{(n)} = \mathbb{E}\eta_{v,k}^{(n)},$$

conditioned on the price process. As a result,

$$\|\eta_v^{(n)}(\cdot,t) - \widetilde{u}^{(n)}(\cdot,t)\|_{L^2} = \| \sum_{k=0}^{\lfloor \frac{t}{\Delta t^{(n)}} \rfloor} \left(\mathcal{D}_{v,k}^{(n)}(\cdot,\cdot) - \mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)})] \right) \|_{L^2}.$$

In order to establish convergence of the sum to zero uniformly in time we introduce the L^2-valued triangular martingale-difference-array

$$Y_k^n := \mathcal{D}_{v,k}^{(n)}(\cdot,\cdot) - \mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)})]. \tag{2.3.36}$$

If we can show that there exists $\alpha > \frac{1}{2}$ such that

$$\sup_{n,k} \left(\frac{1}{\Delta t^{(n)}}\right)^{2\alpha} \mathbb{E}[\|Y_k^n\|_{L^2}^2] < \infty, \tag{2.3.37}$$

then Theorem 5.2.8 and Corollary 5.2.9 of the Appendix would guarantee that

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\le m\le \lfloor \frac{T}{\Delta t^{(n)}} \rfloor} \| \sum_{k=0}^{m} Y_k^n \|_{L^2} > \epsilon \right) = 0 \tag{2.3.38}$$

and the proposition would be proved. To establish (2.3.37) it suffices to prove that

$$\begin{aligned} \sup_{n,k} \left(\frac{1}{\Delta t^{(n)}}\right)^{2\alpha} \mathbb{E}[\|\mathcal{D}_{v,k}^{(n)}(\cdot,\cdot)\|_{L^2}^2] &< \infty \\ \sup_{n,k} \left(\frac{1}{\Delta t^{(n)}}\right)^{2\alpha} \mathbb{E}[\|\mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)}, \eta_{v,k}^{(n)})]\|_{L^2}^2] &< \infty. \end{aligned}$$

For the second inequality, we notice that

$$\begin{aligned}\mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)},\eta_{v,k}^{(n)})] &= \Delta p^{(n)}\cdot p_k^A\cdot[T_+^{(n)}(\eta_{v,k}^{(n)})-\eta_{v,k}^{(n)}]+\Delta p^{(n)}\cdot p_k^B\cdot[T_-^{(n)}(\eta_{v,k}^{(n)})-\eta_{v,k}^{(n)}]\\ &\quad +(1-\Delta p^{(n)})\cdot\Delta v^{(n)}\cdot\left\{-p_k^C\cdot f_k^{(n),C}\cdot\eta_{v,k}^{(n)}+p_k^D\cdot f_k^{(n),D}\right\}.\end{aligned}$$

Using the isometry property of the translation operators and Assumption 2.3.4 we find generic constants $C_0, C_1 < \infty$ such that almost surely

$$\|\mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)},\eta_{v,k}^{(n)})]\|_{L^2}^2 \;\leq\; C_0\left(\Delta p^{(n)}\right)^2\|\eta_{v,k}\|_{L^2}^2+C_1\left(\Delta v^{(n)}\right)^2.$$

Taking expectations and using the bound on the expected squared L^2-norm of the volume density functions established in Lemma 5.2.10 of the Appendix it follows from Assumption 2.3.7 that there there exists a generic constant $C_0 < \infty$ such that:

$$\begin{aligned}\left(\frac{1}{\Delta t^{(n)}}\right)^{2\alpha}\mathbb{E}[\|\mathbb{E}[\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)},\eta_{v,k}^{(n)})]\|_{L^2}^2 &\leq \left(\frac{1}{\Delta t^{(n)}}\right)^{2\alpha}\left[C_0\left(\Delta p^{(n)}\right)^2+C_0\left(\Delta v^{(n)}\right)^2\right]\\ &\leq C_0\left(\frac{\Delta p^{(n)}}{\left(\Delta t^{(n)}\right)^{\alpha}}\right)^2+o(1)\\ &\leq C_0.\end{aligned}$$

Conditioning on the realization of prices and volumes we see that

$$\begin{aligned}&\left(\frac{1}{\Delta t^{(n)}}\right)^{2\alpha}\mathbb{E}[\|\mathcal{D}_{v,k}^{(n)}(\cdot,\cdot)\|_{L^2}^2]\\ &=\left(\frac{1}{\Delta t^{(n)}}\right)^{2\alpha}\mathbb{E}\left[\mathbb{E}\left[\|\mathcal{D}_{v,k}^{(n)}(\eta_{\gamma,k}^{(n)},\eta_{v,k}^{(n)})\|_{L^2}^2|\eta_{v,k}^{(n)},\eta_{\gamma,k}^{(n)}\right]\right]\leq C_0.\end{aligned}$$

This also proves the first inequality above and hence the assertion. □

2.4 Conclusion and Outlook

For a random discrete order book model in which the prices drive the dynamics, we were able to show a macroscopic limit in the sense of strong and weak laws of large numbers, respectively. Our scalings introduced a slower time scale for the price dynamics and a faster time scale, which may be qualitatively motivated by empirical studies e.g. by Hautsch and Huang [42]. The limiting models are deterministic and as such, they serve as first order approximations of the proposed order book models and could be interpreted as the *drifts* of the order books. Applications could include estimating the time-to-fill in the original model i.e. the time one has to wait until a placed limit order is filled/executed. A simple approximation is given by the hitting time of the limit price of the placed order, using only the limiting price process. A refinement would be to consider the volume placed (over the original price tick interval) by calculating the time-to-fill as the time it takes, until "placed limit order volume + the total limit volume in front of the order" becomes zero. The fact that PDE-solutions of the volume densities can be given in closed form for the limiting model, facilitates these calculations. Also, simply adding an appropriate stochastic process as the *noise*, yields a credible *ad hoc model* including random fluctuations. Since the limiting model is smooth, one could use it to study optimal trading strategies.

Considering a more generalized state-dependence, one may let the order flow in the order book be dependent on the best bid and ask prices as well as on the standing buy and sell volume density. Again, we want to make a smooth approximation in the sense of a law of large numbers. For such a limiting model, the standing buy and sell volume density would be the solution of the coupled first order quasi-linear hyperbolic PDE:s:

$$\begin{cases} v_t(x,t) = \frac{1}{m(\gamma(t))}\Big(A\left(\gamma(t)\right)v_x(x,t) + B(v(x,t),x,\gamma(t))v(x,t) + c\left(v(x,t),x,\gamma(t)\right)\Big), \\ v(x,0) = v_0(x), \quad x \in \mathbb{R} \end{cases} \tag{2.4.1}$$

provided that a sufficiently smooth solution of (2.4.1) exists and several additional assumptions on the smoothness of the state parameters hold. Below, we will state these conditions more precisely, relying on the classical result by Strang [74] on the method of local linearization of the finite difference approximation, relating to systems of first order quasi-linear hyperbolic PDE:s. The result holds for the following initial-value problem.

Let $x \in \mathbb{R}$, find $u(x,t) = (u_1(x,t), \dots, u_l(x,t))'$ such that

$$\begin{cases} u_i(x,t) & = \mathcal{A}(u,x,t)\frac{\partial u_i(x,t)}{\partial x} + \mathfrak{a}(u,x,t) \quad \text{for } i = 1,\dots,l \text{ and } t \in [0,T] \\ u(x,0) & = u_0(x), \end{cases} \tag{2.4.2}$$

where $\mathcal{A}$ and $\mathfrak{a}$ are operators describing the dynamics of the LOB.

Definition 2.4.1 (First variation of finite difference operator)**.** *Let*

$$\mathcal{H}^{(n)}(u_k^{(n)}, x, t) = \mathcal{H}^{(n)}\left(u_k^{(n)}(x - q\Delta x^{(n)}), \dots, u_k^{(n)}(x + q\Delta x^{(n)}), x, t\right) \tag{2.4.3}$$

be a finite difference operator for the PDE-System (2.4.2). The first variation of $\mathcal{H}^{(n)}$ is given by the linear difference operator

$$M^{(n)}(u_k^{(n)}) := \sum_{j=-q}^{q} C_j(u_k^{(n)})u_k^{(n)}(x + j\Delta x^{(n)}), \tag{2.4.4}$$

where

$$C_j(u_k^{(n)}) := \frac{\partial \mathcal{H}^{(n)}}{\partial \underline{a}_j}\left(u_k^{(n)}, \dots, u_k^{(n)}, x, t\right) \tag{2.4.5}$$

denotes the Jacobian of the vector valued operator $\mathcal{H}^{(n)}\left(\underline{a}_{-q}, \dots, \underline{a}_q, x, t\right)$ *with respect to the vector valued variable* $\underline{a}_j$*, evaluated at* $\underline{a}_{-q}, \dots, \underline{a}_q = u_k^{(n)}$.

Definition 2.4.2 (L^2-stability)**.** *The finite difference approximation* $u^{(n)}$*, is said to be* L^2*-stable if its value after* k *steps is bounded by a constant multiple of the* L^2*-norm of the initial data:*

$$\|u_k^{(n)}\|_{L^2}^2 \leq K \cdot \|u_0^{(n)}\|_{L^2}^2. \tag{2.4.6}$$

Theorem 2.4.3 (Strang, [74, Theorem I on p.40])**.** *Suppose that the finite difference operator* $\mathcal{H}^{(n)}$ *is consistent with order of accuracy* p *and that its first variation is* L^2*-stable. Additionally, suppose that* $\mathcal{A}$*,* $\mathfrak{a}$*,* $\mathcal{H}^{(n)}$ *and the solution* u *of the initial value problem (2.4.2) are* C^{p+q_0+3} *in the domain*

$$D_0 := \{(x,t) | t \in [0, t_0], \frac{\Delta x}{\Delta t}\frac{(x - x_0)}{t - t_0} \in L_h\} \tag{2.4.7}$$

where L_h *is the convex hull of* $\{j \in (-q, \dots, q)\}$ *in (2.4.4) and* q_0 *denotes the dependence on the inhomogeneous term in the Taylor expansion of* $\mathcal{H}^{(n)}$.

Then, it holds that

$$\left| u^{(n)}(x_0, t_0) - u(x_0, t_0) \right| = \mathcal{O}\left((\Delta t^{(n)})^p \right) \quad \text{for } (x_0, t_0) \in D_0, \tag{2.4.8}$$

where $u^{(n)}$ *is the finite difference approximation generated by* $\mathcal{H}^{(n)}$.

A crucial part is to establish that at least a local solution exists for the system (2.4.2). Given smooth regularity of $\mathcal{A}$ and $\mathfrak{a}$, a local solution exists if the coefficients only depend on the densities, i.e. for the quasi-linear hyperbolic system of scalar first-order PDEs:

$$\begin{cases} \frac{\partial}{\partial t} u_1(x,t) + a_1(u_1(x,t), u_2(x,t)) \frac{\partial}{\partial x} u_1(x,t) &= b_1(u_1(x,t), u_2(x,t)) \\ \frac{\partial}{\partial t} u_2(x,t) + a_2(u_1(x,t), u_2(x,t)) \frac{\partial}{\partial x} u_2(x,t) &= b_2(u_1(x,t), u_2(x,t)) \\ \\ u_1(x,0) = u_{1,0}(x) \\ u_2(x,0) = u_{2,0}(x) \end{cases} \tag{2.4.9}$$

In matrix notation, we write (2.4.9) as

$$\begin{cases} \frac{\partial}{\partial t}u + A(u)\frac{\partial}{\partial x}u = b(u), \\ u(x,0) = u_0(x) \end{cases} \tag{2.4.10}$$

where $u := \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}$, $A(u) := \begin{pmatrix} a_1(u) & 0 \\ 0 & a_2(u) \end{pmatrix}$ and $b(u) := \begin{pmatrix} b_1(u) \\ b_2(u) \end{pmatrix}$.

Now, let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$ containing 0. We assume that

- a_1, a_2, b_1, b_2 vanish at 0 and that they are smooth and bounded functions of $u \in \mathcal{U}$:

$$a_1, a_2, b_1, b_2 \in C^\infty(\mathcal{U}, [-c, c]), \quad c \in (0, \infty), \tag{2.4.11}$$

- the initial data is smooth, non-negative and bounded:

$$u_{1,0}(x), u_{2,0}(x) \in C^\infty(\mathbb{R}, [0, d]), \quad d \in (0, \infty). \tag{2.4.12}$$

Under these assumptions a local solution exists, i.e. one has that there exists a $T > 0$ such that (2.4.9) i.e. (2.4.10) has a unique classical solution $u \in C^1(\mathbb{R} \times [0,T], \mathbb{R}^2)$ with $u(x,0) = u_0(x)$ (see Benzoni-Gavage and Serre [10, Theorem 10.1,p.293]).

Global solutions are, however, a different matter and typically hold under quite strong assumptions (see e.g. Li [56]). It seems that to prove laws of large numbers, for fully coupled volume densities, much additional work needs to be done.

3 Functional Central Limit Theorems for Limit Order Books

In the first section of this chapter we prove a functional central limit theorem for a simple model of an order book. Under appropriate scaling and given technical conditions on the order flow, the bid and ask prices converge weakly to a semi-martingale reflecting Brownian motion in the set of admissible prices and the relative volume densities to the modulus of a two parameter Brownian motion, where the first parameter is the relative price distance and the second parameter is time. The second section provides a simple derivation of an SPDE for the model considered in Chapter 2, under a strong stationarity assumption for the limit order placements/fluctuations.

3.1 A Simple LOB Model with Spread Trading

We define a sequence of discrete LOB models, where the model with $n = 1$ corresponds to an original observable model. The dynamics are considered in event time and the inter arrival times are assumed to be deterministic[1] and equidistant for fixed n.

We assume that there are two types of dynamics involved in placing the active orders, which lead to price changes. Firstly, there is a zero-intelligence type of placement which is random and independent of the current state. These orders move either of the best bid and ask price (one price tick up or down). Additionally, there is a more strategic type of activity during certain states of the order book.

Assumption 3.1.1 (Existence of minimum and spread trading)**.** *We assume that*

i) There always exist buyers who are willing to buy the security at the lowest possible price of one price tick $p = \Delta x^{(n)}$.

ii) As soon as the spread is at its possible minimum, i.e. when the difference between the best bid and ask price is exactly one price tick, there always exist agents who submit market buy and sell orders for the the quantity available at the best bid and ask price, simultaneously.

[1] Random inter arrival times may be introduced quite easily, see the conclusion at the end of the chapter.

To simplify the dynamics, we assume that order placement occurs independently of the current state of the model. Specifically, limit order placement occurs relative to the best bid and ask price. As we allow cancelation and want non-negativity of the standing limit order volumes, the variables for the standing volumes are modeled to be the modulus of the cumulative limit order fluctuations. The latter are conveniently modeled using random fields. Volumes below [above] the best bid [ask] price are considered as standing buy [sell] limit order volumes. The spread is again modeled using a shadow book, similar to what was done in Chapter 2. We do not model the coupling of prices and orders explicitly, but we state sufficient assumptions for the functional central limit theorem to hold and discuss how these could be weakened at the end of the chapter.

3.1.1 Model Dynamics

Again, we have events - labelled A, ..., H - that change the state of the book:

$\mathbf{A} := \{\text{market sell order}\}$ $\quad$ $\mathbf{B} := \{\text{buy limit order placed in the spread}\}$

$\mathbf{C} := \{\text{cancelation of buy volume}\}$ $\quad$ $\mathbf{D} := \{\text{buy limit order not placed in spread}\}$

$\mathbf{E} := \{\text{market buy order}\}$ $\quad$ $\mathbf{F} := \{\text{sell limit order placed in the spread}\}$

$\mathbf{G} := \{\text{cancelation of sell volume}\}$ $\quad$ $\mathbf{H} := \{\text{sell limit order not placed in the spread}\}$.

We describe the state dynamics by a stochastic process $\{\hat{S}_k^{(n)}\}_{k\in\mathbb{N}}$ such that

$$\hat{S}_k^{(n)} := \left(\hat{P}_k^{(n)}, \hat{v}_{b,k}^{(n)}, \hat{v}_{s,k}^{(n)}\right)', \tag{3.1.1}$$

where the first component is the two-dimensional price process $\hat{P}_k^{(n)} := \left(\hat{B}_k^{(n)}, \hat{A}_k^{(n)}\right)'$, such that $\hat{B}_k^{(n)}$ and $\hat{A}_k^{(n)}$ are the best bid and ask price after k events in the n:th model, respectively. The second and third component of (3.1.1), $\hat{v}_{b,k}^{(n)}$ and $\hat{v}_{s,k}^{(n)}$ refer to the relative buy [sell] volume of the book (visible/standing volume and shadow book), in the $n : th$ model, respectively.

The state dynamics of the model are again, defined recursively i.e.

$$\hat{S}_0^{(n)} = \hat{s}_0^{(n)} \quad \text{and} \quad \hat{S}_{k+1}^{(n)} = \hat{S}_k^{(n)} + \hat{\mathcal{D}}_k^{(n)}\left(\hat{S}_k^{(n)}\right), \quad k \geq 0. \tag{3.1.2}$$

For the state space, we denote $E := \mathbb{R}_+^2 \times \mathbb{R}_+^{\mathbb{Z}^2} \times \mathbb{R}_+^{\mathbb{Z}^2}$. On the other hand, the price process will live in a discrete set of admissible prices for every n.

Definition 3.1.2 (Set of admissible prices in the n:th and limiting model)**.** *The set of admissible prices $G^{(n)}$ for the price process $\hat{P}_k^{(n)} := \left(\hat{B}_k^{(n)}, \hat{A}_k^{(n)}\right)'$ for all $k \geq 0$ in the* *n:th model is given by*

$$G^{(n)} := \left\{x, y \in \{\Delta x^{(n)}, 2\Delta x^{(n)}, \ldots\} \mid x \leq y + \Delta x^{(n)}\right\} \tag{3.1.3}$$

and for the boundary, we denote

$$\partial G^{(n)} := \left\{x, y \in G^{(n)} \mid x = \Delta x^{(n)}\right\} \bigcup \left\{x, y \in G^{(n)} \mid x = y + \Delta x^{(n)}\right\}. \tag{3.1.4}$$

In the limiting model, i.e. when the price tick has converged to zero, we denote the admissible prices as

$$G = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq y\right\}, \tag{3.1.5}$$

and its boundary as

$$\partial G := \{(x, y) \in G \mid x = 0\} \bigcup \{(x, y) \in G \mid y = x\}. \tag{3.1.6}$$

We continue and define the dynamics of the various order types.

Active orders

Market orders and placements of limit orders in the spread lead to price changes[2]. Just as in Chapter 2 we refer to these order types as *active orders*. Again, we assume that market orders match precisely against the standing volume at the best bid price and that all buy limit orders placed in the spread improve prices by one tick. We have to consider what happens when the best bid and ask price approach each other and the minimum price, respectively. To guarantee admissible prices i.e. that $\hat{P}_k^{(n)} \in G^{(n)}$ holds for all event times $k \geq 0$, we introduce natural reflecting conditions. To this end, we define the price process as the superposition of a random walk in the interior and a reflection process, coherent with Assumption 3.1.1 which only contributes on the boundaries.

Thus, if the k:th event is a market sell order (Event $\mathbf{A}$), the best bid price decreases by one tick and if the k:th event is a buy limit order placed in the spread (Event $\mathbf{B}$), the best bid price increases by one tick. If the spread is one tick, there is again reflection and the best bid price decreases by one tick:

$$\hat{B}_{k+1}^{(n)} = \hat{B}_k^{(n)} - \Delta x^{(n)} \mathbb{1}_k^A + \Delta x^{(n)} \mathbb{1}_k^B + \Delta x^{(n)} \mathbb{1}_{\left\{\hat{B}_k^{(n)} = \Delta x^{(n)}\right\}} - \Delta x^{(n)} \mathbb{1}_{\left\{\hat{A}_k^{(n)} - \hat{B}_k^{(n)} = \Delta x^{(n)}\right\}}.$$

Similarly, for the best ask price, we have that if the k:th event is a market buy order (Event $\mathbf{E}$), the best ask price increases by one tick and if the k:th event is a sell limit order placed in the spread (Event $\mathbf{F}$), the best ask price decreases by one tick. If the

[2] A market order that does not lead to a price change can be viewed as a cancelation of standing volume at the best bid/ask price.

spread is one tick, there is again reflection and the best ask price increases by one tick:

$$\hat{A}_{k+1}^{(n)} = \hat{A}_k^{(n)} + \Delta x^{(n)} \mathbb{1}_k^E - \Delta x^{(n)} \mathbb{1}_k^F + \Delta x^{(n)} \mathbb{1}_{\left\{\hat{A}_k^{(n)} - \hat{B}_k^{(n)} = \Delta x^{(n)}\right\}}.$$

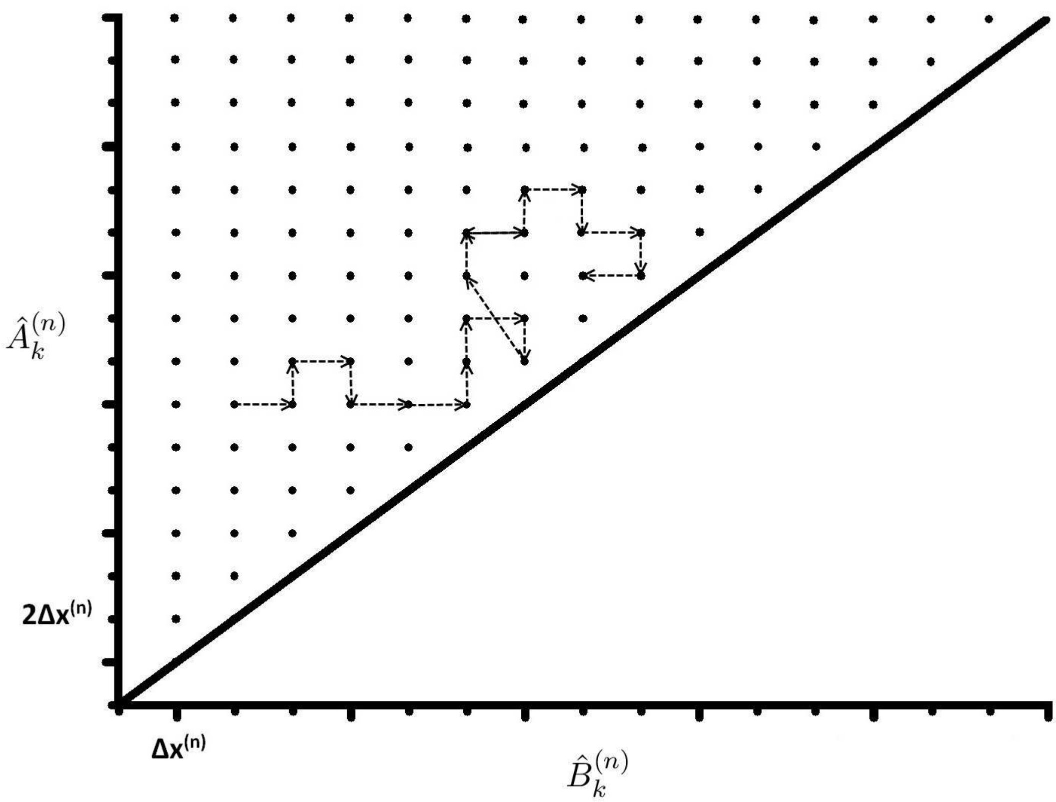

Figure 3.1: A possible path of the price process $\hat{P}_k^{(n)} = (\hat{B}_k^{(n)}, \hat{A}_k^{(n)})'$. Away from the boundaries it is an unbiased random walk on the 2-D-Lattice of price ticks and at the boundaries there is reflection toward the interior of the admissible price domain.

Passive orders

Passive orders are limit order placements outside the spread and cancelations of standing volume. These orders correspond to events $I = C, D, G, H$ and do not effect the best bid and ask prices directly. As already discussed, these events typically happen more often than active orders i.e. events that change prices. We will model the volume density close to the spread and assume that it is block-shaped but that the height of the blocks may vary randomly. Another simplifying assumption is that the random fluctuations are invariant when the price changes i.e. translations of the random densities occur. It is very convenient to model the height fluctuations of the buy [sell] volume densities as a random field[3] that is *probabilistically invariant* under back and forth translations due to active orders over time as well as price levels. To specify our assumptions in detail, we need some further definitions.

Definition 3.1.3 (Random field fluctuations)**.** *The underlying fluctuations of the buy [sell] volume density is given by the real valued random field*

$$\left\{\xi_{b,k}^{j}\right\}_{j,k\in\mathbb{Z}} \quad \left[\left\{\xi_{s,k}^{j}\right\}_{j,k\in\mathbb{Z}}\right],$$

where $\xi_{b,k}^{j}$ $[\xi_{s,k}^{j}]$ denotes the buy [sell] volume density fluctuation at j levels away from the spread after k events due to a cancelation (event C $[G]$) or limit order placement (event D $[H]$). As before, price levels $j < 0$ describe the shadow book i.e. the volumes in the spread at price levels $j \geq 0$ describe the standing visible volume in the order book.

Remark 3.1.4. *The variables $\xi_{b,k}^{j}$ $[\xi_{s,k}^{j}]$, do not model the buy [sell] volume density levels per se. These random field fluctuations are used as a mathematical tool for the actual volume density modeling as defined in (3.1.9) of Assumption 3.1.7.*

The active orders make the relative volumes shift/translate back and forth and to describe these shifts, we will again introduce a translation operator. In contrast to Definition 2.1.2 of Chapter 2, we will assume that the fluctuations, given in 3.1.3 are ergodic and thus the translation operator will also operate on event times.

Definition 3.1.5 (The translation operator $\tau_{l,m}$)**.** *Let $\{X(j,k)\}_{j,k\in\mathbb{Z}}$ be a two parameter stochastic process. The translation operator $\tau_{l,m}$ with shift parameters $l, m \in \mathbb{Z}$ acting on $\mathbb{R}^{\mathbb{Z}^2}$ is given by*

$$(\tau_{l,m}X)(j,k) = X(j-l, k-m), \quad j,k \in \mathbb{Z}. \tag{3.1.7}$$

Now, we are able to specify our central assumption.

Assumption 3.1.6 (Ergodicity and martingale difference field assumption for the random field)**.** *The buy [sell] volume density fluctuations after k events at price level $j\Delta x$ are given by a real valued random field $\{\xi_{b,k}^{j}\}_{j,k\in\mathbb{Z}}$ $[\{\xi_{s,k}^{j}\}_{j,k\in\mathbb{Z}}]$ which is*

[3] In Chapter 2, the model of the volume densities is in effect also a random field model i.e. the volumes are a random function defined over some Euclidean space, see Adler [2, p.1].

i) Translation invariant:

$$\mathbb{P}(\tau_{l,m}(A)) = \mathbb{P}(A), \quad \textit{for } A \in \mathcal{F} \textit{ and } l, m \in \mathbb{Z}.$$

ii) Ergodic: $\mathbb{P}$ *is trivial on the* σ*-algebra of translation invariant subsets i.e. either* $\mathbb{P}(A) = 0$ *or* $\mathbb{P}(A) = 1$ *for all* $A \in \mathcal{I}$*, where*

$$\mathcal{I} := \{A \in \mathcal{F} : \tau_{l,m}(A) = A\}$$

is the σ*-algebra of invariant subsets of* $\mathbb{R}^{\mathbb{Z}^2}$.

iii) A martingale difference field:

$$\mathbb{E}\left[\xi_{b,k+1}^{j+1} \middle| \mathcal{P}_{b,k}^{j}\right] = 0, \quad \textit{where} \quad \mathcal{P}_{b,k}^{j} := \sigma\left(\xi_{b,m}^{l} : m \leq k \vee l \leq j\right)$$

$$\left[\mathbb{E}\left[\xi_{s,k+1}^{j+1} \middle| \mathcal{P}_{s,k}^{j}\right] = 0, \quad \textit{where} \quad \mathcal{P}_{s,k}^{j} := \sigma\left(\xi_{s,m}^{l} : m \leq k \vee l \leq j\right)\right]$$

is the σ*-algebra generated by the buy [sell] volume density fluctuations.*

3.1.2 Scaling Limits and Main Result

The main result yields a mesoscopic approximation in the unit event time and relative price interval. Before we state our result, we make the following scaling assumptions.

Assumption 3.1.7 (Scaling assumptions for the Functional Central Limit Theorem)**.** *Assume that:*

- *The price tick is scaled by* $\Delta x^{(n)} := \frac{\Delta x}{\sqrt{n}}$*, the event time is scaled by* n *(which corresponds to the time tick scaling* $\Delta t^{(n)} := \frac{1}{n}$*) and the order book densities fluctuations by* $\frac{1}{n}$ *(this corresponds to a limit order/cancelation volume scaling of* $\frac{1}{n^{3/2}}$ *since* $\frac{\Delta v^{(n)}}{\Delta x^{(n)}} = \frac{1}{n} \Leftrightarrow \Delta v^{(n)} = \frac{1}{n^{3/2}}$*).*
- *The event indicator variables for the active orders* $\mathbb{1}_k^A, \mathbb{1}_k^B, \mathbb{1}_k^E, \mathbb{1}_k^F$ *are categorically distributed, state-independent and pairwise independent. For the passive orders, the buy [sell] events are implicitly defined over the random field* $\{\xi_{b,k}^j\}_{j,k\in\mathbb{Z}}$ $[\{\xi_{s,k}^j\}_{j,k\in\mathbb{Z}}]$ *in Definition 3.1.6.*
- *For the probability of active order events* $I = A, B, E, F$*, i.e.* $p^I = \mathbb{E}\left[\mathbb{1}_k^I\right]$ *we assume that the probability of a market sell [buy] order is equal to the probability of a buy [sell] order placed into the spread:*

$$p^A = p^B \quad \textit{and} \quad p^E = p^F. \tag{3.1.8}$$

- *The initial best bid and ask prices in the n : th model converge to the initial prices of the limiting model:*

$$\left(\hat{B}_0^{(n)}, \hat{A}_0^{(n)}\right)' \to \left(\hat{B}_0, \hat{A}_0\right)' \quad \textit{in probability as } n \to \infty$$

such that almost surely, none of the the initial bid and ask prices are on the boundary of the admissible price region:

$$\left(\hat{B}_0^{(n)}, \hat{A}_0^{n}\right)' \notin \partial G^{(n)} \textit{ for all } n \quad \textit{and} \quad \left(\hat{B}_0, \hat{A}_0\right)' \notin \partial G \quad \textit{a.s.}$$

- *The standing relative limit buy [sell] volume densities are given by*

$$\hat{v}_b^{(n)}(x,t) := \left| \frac{1}{\sigma_b n} \sum_{k=0}^{\lfloor nt \rfloor} \sum_{j=0}^{\lfloor nx \rfloor} \xi_{b,k}^j \right|, \quad x,t \in [0,1]$$

$$\left[\hat{v}_s^{(n)}(x,t) := \left| \frac{1}{\sigma_s n} \sum_{k=0}^{\lfloor nt \rfloor} \sum_{j=0}^{\lfloor nx \rfloor} \xi_{s,k}^j \right|, \quad x,t \in [0,1] \right], \tag{3.1.9}$$

where $\sigma_b^2 = \mathbb{E}\left[(\xi_{b,0}^0)^2\right] > 0$ $\left[\sigma_s^2 = \mathbb{E}\left[(\xi_{s,0}^0)^2\right] > 0\right]$ *is the variance of the initial buy [sell] volume density.*

The buy [sell] side of the spread volumes is given by the shadow book

$$\hat{v}_b^{(n)}(x,t) := \left| \frac{1}{\sigma_b n} \sum_{k=0}^{\lfloor nt \rfloor} \sum_{j=\lfloor nx \rfloor}^{0} \xi_{b,k}^j \right|, \quad x \in [-1,0], t \in [0,1]$$

$$\left[\hat{v}_s^{(n)}(x,t) := \left| \frac{1}{\sigma_s n} \sum_{k=0}^{\lfloor nt \rfloor} \sum_{j=\lfloor nx \rfloor}^{0} \xi_{s,k}^j \right|, \quad x \in [-1,0], t \in [0,1] \right]. \tag{3.1.10}$$

Observing the state of the model in continuous time and over continuous relative prices, we introduce the process

$$\begin{aligned} \hat{S}^{(n)}(\cdot,t) &:= \left(\hat{B}^{(n)}(t), \hat{A}^{(n)}(t), \hat{v}_b^{(n)}(\cdot,t), \hat{v}_s^{(n)}(\cdot,t)\right)' \\ &= \left(\hat{B}_k^{(n)}, \hat{A}_k^{(n)}, \hat{v}_{b,k}^{(n)}, \hat{v}_{s,k}^{(n)}\right)', \quad \text{for } t \in [t_k^{(n)}, t_{k+1}^{(n)}). \end{aligned} \tag{3.1.11}$$

Thus, for $t \in [0,1]$ we have:

$$\begin{cases} \hat{B}^{(n)}(t) & = \hat{B}^{(n)}(0) + \sum_{k=0}^{\lfloor nt \rfloor} \Delta \hat{B}_k^{(n)} \\ \\ \Delta \hat{B}_k^{(n)} & := \Delta x^{(n)} \left(-\mathbb{1}_k^A + \mathbb{1}_k^B + \mathbb{1}_{\left\{ \hat{B}_k^{(n)} = \Delta x^{(n)} \right\}} - \mathbb{1}_{\left\{ \hat{A}_k^{(n)} - \hat{B}_k^{(n)} = \Delta x^{(n)} \right\}} \right) \end{cases} \tag{3.1.12}$$

and

$$\begin{cases} \hat{A}^{(n)}(t) & = \hat{A}^{(n)}(0) + \sum_{i=0}^{\lfloor nt \rfloor} \Delta \hat{A}_k^{(n)} \\ \\ \Delta \hat{A}_k^{(n)} & := \Delta x^{(n)} \left(\mathbb{1}_k^E - \mathbb{1}_k^F + \mathbb{1}_{\left\{ \hat{A}_k^{(n)} - \hat{B}_k^{(n)} = \Delta x^{(n)} \right\}} \right) \end{cases} \tag{3.1.13}$$

for the best bid and ask price, respectively.

The limiting price process will be a semi-martingale reflecting Brownian motion, a stochastic process which acts as a Brownian motion while inside of the domain and is instantaneously pushed back at its boundaries. It appears as a limit in various multi-class queueing systems, see e.g. Kruk [53] and the survey by Williams [77].

Definition 3.1.8 (Semimartingale reflecting Brownian motion, see Kang and Williams [51, Def. 2.1, p.741])**.** *A Semimartingale Reflecting Brownian Motion (SRBM) associated with the data $(G, \mu, \Gamma, \{\gamma_i, i \in \mathcal{I}\}, \nu)$ is an $\mathcal{F}_t$-adapted, d-dimensional process W defined on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ such that*

i) $\mathbb{P}$*-a.s.* $W(t) = X(t) + \sum_{i \in \mathcal{I}} \int_{(0,t]} \gamma^i \left(W(s)\right)' \mathrm{d}Y_i(s)$ *for all* $t \geq 0$*,*

ii) $\mathbb{P}$*-a.s.* W *has continuous paths and* $W(t) \in \overline{G}$ *for all* $t \geq 0$*,*

iii) X *is a d-dimensional Brownian motion with drift vector* μ*, covariance matrix* Γ *and initial distribution* ν*,*

iv) For each $i \in \mathcal{I}$*,* Y_i *is an* $\mathcal{F}_t$*-adapted, one-dimensional process such that* $\mathbb{P}$*-a.s.*

a) $Y_i(0) = 0$*,*

b) Y_i *is continuous and nondecreasing,*

c) $Y_i(t) = \int_{(0,t]} \mathbb{1}_{\{W(s) \in \partial G_i \cap \partial G\}} \mathrm{d}Y_i(s)$ *for all* $t \geq 0$*.*

The relative volume densities will converge to the modulus of a 2-parameter Brownian motion which is also known as a Brownian sheet, see Khoshnevisan [52, p.147]. It has other equivalent definitions which can be derived from the general definition of a white noise, see Definition 5.3.2 in the Appendix. The definition below generalizes standard Brownian motion to two parameters and is very practical for the method of proof employed in Poghosyan and Rœlly [64, Theorem 3 on p.241].

Definition 3.1.9 (2-parameter Brownian motion, Brownian sheet)**.** *The real-valued stochastic process* $W(x,t) : \Omega \times ([0,1] \times [0,1]) \to \mathbb{R}$ *such that*

i) $\mathbb{P}(W(x,t) = 0) = 1$*, if* $x = 0$ *or* $t = 0$.

ii) For pairwise disjoint blocks

$$B_1 := (x_1^l, x_1^r] \times (t_1^l, t_1^r], \ldots, B_k := (x_k^l, x_k^r] \times (t_k^l, t_k^r]$$

with $x_i^l, x_i^r, t_i^l, t_i^r \in [0,1]$*,* $i = 1, \ldots, k$ *the increments*

$$W(B_i) := \left(W(x_i^r, t_i^r) - W(x_i^r, t_1^l)\right) - \left(W(x_i^l, t_i^r) - W(x_i^l, t_i^l)\right)$$

are independent normal random variables with

$$\mathbb{E}[W(B_1)] = \ldots = \mathbb{E}[W(B_k)] = 0$$

and the variances are given by the areas of the blocks:

$$Var(W(B_1)) = (x_1^r - x_1^l) \cdot (t_1^r - t_1^l), \ldots, Var(W(B_k)) = (x_k^r - x_k^l) \cdot (t_k^r - t_k^l).$$

iii) The paths of W *are continuous a.s.*

is called a standard 2-parameter Brownian motion.

The main result of the section is the following theorem.

Theorem 3.1.10 (Functional Central Limit Theorem for a simple order book model)**.** *If Assumptions 3.1.6 and 3.1.7 hold, we have that*

$$\hat{P}^{(n)}(t) = \left(\hat{B}^{(n)}(t), \hat{A}^{(n)}(t)\right)' \Rightarrow \hat{W}(t) \quad \text{as } n \to \infty \tag{3.1.14}$$

where $\hat{W}(t)$ *is a two-dimensional semi-martingale reflecting Brownian motion associated with the data* $(G, \mu, \Gamma, \{\gamma_i, i \in \mathcal{I}\}, \nu)$*, where the set of admissible prices is given by*

$$G = \left\{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le y\right\}, \tag{3.1.15}$$

the drift vector is zero i.e. $\mu = \mathbf{0}$*, the covariance matrix is given by*

$$\Gamma := \Delta x^2 \begin{pmatrix} p^A + p^B & 0 \\ 0 & p^E + p^F \end{pmatrix}. \tag{3.1.16}$$

and the boundaries of G *in (3.1.15)*

$$\partial G_1 := \left\{ (x, y) \in \mathbb{R}_+^2 \mid x = 0 \right\} \quad \textit{and} \quad \partial G_2 := \left\{ (x, y) \in \mathbb{R}_+^2 \mid y = x \right\}$$

have constant reflection vectors

$$\gamma^1 := (1, 0) \quad \textit{and} \quad \gamma^2 := \frac{1}{\sqrt{2}} (-1, 1)\,.$$

respectively. The initial distribution ν *is singular i.e.*

$$\nu\Big(\{ (\hat{B}(0), \hat{A}(0))' = (\hat{B}_0, \hat{A}_0)' \in G \setminus \partial G \} \Big) = 1. \tag{3.1.17}$$

Let the sequence of order book models be zero intelligence models, then the standing buy [sell] volume density

$$v_b^{(n)}(x,t) \Rightarrow |W_b(x,t)| \quad \left[v_s^{(n)}(x,t) \Rightarrow |W_s(x,t)| \right], \quad \textit{as } n \to \infty \tag{3.1.18}$$

where $W_b(x,t)$ $[W_s(x,t)]$ *is a 2-parameter Brownian motion and* $(x,t) \in [0,1] \times [0,1]$*.*

As a corollary to the main result above, we get the following approximation for absolute prices close to the spread.

Corollary 3.1.11 (Time evolution of standing limit order volume close to the spread)**.** *For the limit model, the observable standing limit order density at time* t *is given by*

$$w(p,t) := \left| W_b\left(\hat{B}(t) - p, t \right) \right| \mathbb{1}_{[\hat{B}(t)-1, \hat{B}(t))}(p) + \left| W_s\left(p - \hat{A}(t), t \right) \right| \mathbb{1}_{[\hat{A}(t), \hat{A}(t)+1)}(p),$$

where $p \in [\hat{B}(t) - 1, \hat{A}(t) + 1]$ *denotes the price of the security,*

the best bid $[ask]$ *price* $\hat{B}$ $[\hat{A}]$ *is the* x $[y]$ *component of the SRBM* $\hat{W}$ *in (3.1.14) and*

W_b $[W_s]$ *is a 2-parameter Brownian motion for* $(\hat{B}(t) - p) \in [0,1]$ $\Big[(p - \hat{A}(t)) \in [0,1]\Big]$ *and equal to 0 for* $(\hat{B}(t) - p) \notin [0,1]$ $\Big[(p - \hat{A}(t)) \notin [0,1]\Big]$*.*

3.1.3 Proof of the Main Result

We use the general invariance principle for convex polyhedrons by Kang and Williams (Theorem 3.1.14) to prove the convergence of the price process by verifying that their sufficient conditions hold for our model.

Convergence of the price process

We now state the results which we will use to prove the price convergence part of Theorem 3.1.10. In the general setting, the SRBM lives on $G := \bigcap_{i\in\mathcal{I}} G_i$ which is assumed to be a nonempty domain in $\mathbb{R}^d$, where $\mathcal{I}$ is a finite index set and for each $i \in \mathcal{I}$, G_i is a nonempty domain in $\mathbb{R}^d$. We assume that the index set $\mathcal{I} \in \{1, 2, \dots, \mathbf{I}\}$, where $\mathbf{I} := |\mathcal{I}|$ is the cardinality of the index set $\mathcal{I}$. For each $i \in \mathcal{I}$, we define a reflection vector $\gamma^i(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$. The drift vector is denoted $\mu \in \mathbb{R}^d$, the $d \times d$ covariance matrix Γ is assumed to be symmetric and strictly positive definite and ν a probability measure on $\left(\overline{G}, \mathcal{B}(\overline{G})\right)$, where $\mathcal{B}(\overline{G})$ denotes the σ-algebra of Borel subsets of the closure $\overline{G}$ of G.

Assumption 3.1.12 (Kang and Williams [51, Assumption 4.1, p.756])**.** *There is a sequence of strictly positive constants $\{\delta^n\}_{n=1}^\infty$ such that for each positive integer n, there are processes W^n, $\widetilde{W}^n$, X^n, α^n having paths in $D([0,\infty), \mathbb{R}^d)$ and processes $Y^n, \widetilde{Y}^n, \beta^n$ having paths in $D([0,\infty), \mathbb{R}^I)$ defined on some probability space $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ such that:*

i) $\mathbb{P}^n$*-a.s.* $W^n = \widetilde{W}^n + \alpha^n$ *and* $\widetilde{W}^n(t) \in U_{\delta^n}(G)$ *for all* $t \geq 0$*,*

ii) $\mathbb{P}^n$*-a.s.* $W^n(t) = X^n(t) + \sum_{i\in\mathcal{I}} \int_{(0,t]} \gamma^{i,n} \left(W^n(s-), W^n(s)\right) \mathrm{d}Y_i^n(s)$ *for all* $t \geq 0$*, where for each* $i \in \mathcal{I}$*,* $\gamma^{i,n} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *is Borel measurable and* $\|\gamma^{i,n}(y,x)\| = 1$ *for all* $x, y \in \mathbb{R}^d$*,*

iii) $Y^n = \widetilde{Y}^n + \beta^n$*, where* β^n *is locally of bounded variation and* $\mathbb{P}^n$*-a.s. for each* $i \in \mathcal{I}$*,*

  a) $\widetilde{Y}_i^n(0) = 0$*,*

  b) $\widetilde{Y}_i^n$ *is nondecreasing and* $\Delta\widetilde{Y}_i^n(t) := \widetilde{Y}_i^n(t) - \widetilde{Y}_i^n(t-) \leq \delta^n$ *for all* $t > 0$*,*

  c) $\widetilde{Y}_i^n(t) = \int_{(0,t]} \mathbb{1}_{\{W^n(s) \in U_{\delta^n}(\partial G_i \cap \partial G)\}} \mathrm{d}\widetilde{Y}_i^n(s)$*.*

iv) $\delta^n \to 0$ *as* $n \to \infty$ *and for each* $\epsilon > 0$*, there is* $\eta_\epsilon > 0$ *and* $n_\epsilon > 0$ *such that for each* $i \in \mathcal{I}$*,* $\|\gamma^{i,n} - \gamma^i(x)\| < \epsilon$ *whenever* $\|x - y\| < \eta_\epsilon$ *and* $n \geq n_\epsilon$*.*

v) $\alpha_n \to \mathbf{0}$ *and the total variation process of* β*,* $\mathcal{V}(\beta^n) \to \mathbf{0}$ *in probability as* $n \to \infty$*.*

vi) $\{X^n\}$ *is* C*-tight.*

To prove the FCLT for bid and ask prices on unbounded price domains in Chapter 3, we will use the convexity property of the set of admissible prices.

Definition 3.1.13 (Convex polyhedron, see Dai and Williams [20])**.** *The domain G is said to be a convex polyhedron, if $\overline{G}$ is defined in terms of $\mathbf{I} \geq 1$ unit vectors $\{n^i, i \in \mathcal{I}\}$ and an $\mathbf{I}$-dimensional vector $\beta = (\beta_1, \dots, \beta_{\mathbf{I}})'$ such that*

$$\overline{G} = \left\{ x \in \mathbb{R}^d \mid \langle n^i, x \rangle \geq \beta_i \text{ for all } i \in \mathcal{I} \right\}. \tag{3.1.19}$$

It is assumed that G is nonempty and that the set $\{(n^1, \beta^1), \dots, (n^{\mathbf{I}}, \beta^{\mathbf{I}})\}$ is minimal in the sense that no proper subset defines $\overline{G}$. For each $i \in \mathcal{I}$, let F_i denote the boundary face: $\{x \in \overline{G} | \langle n^i, x \rangle = \beta_i\}$. Then, n^i is the inward unit normal to F_i.

For domains like those just defined there exist various invariance principles i.e. FCLTs, among others the one below.

Theorem 3.1.14 (Invariance Principle for SRBMs in convex polyhedrons with constant reflection at the boundaries, see Kang and Williams [51, Theorem 5.4, p.773])**.** *Let G be a nonempty domain such that $\overline{G}$ is a convex polyhedron and $\{\gamma^i, i \in \mathcal{I}\}$ be a family of reflection fields of unit length for which there is $a \in (0,1)$ and vector valued functions b, c from ∂G into $\mathbb{R}_+^I$ such that for each $x \in \partial G$,*

a) $\sum_{i \in \mathcal{I}(x)} b_i(x) = 1$ and for each $i \in \mathcal{I}(x)$.

$$b_i(x) \langle n^i(x), \gamma^i(x) \rangle \geq a + \sum_{j \in \mathcal{I}(x) \setminus \{i\}} b_j(x) |\langle n^j(x), \gamma^i(x) \rangle|, \tag{3.1.20}$$

b) $\sum_{i \in \mathcal{I}(x)} c_i(x) = 1$ and for each $i \in \mathcal{I}(x)$,

$$c_i(x) \langle \gamma^i(x), n^i(x) \rangle \geq a + \sum_{j \in \mathcal{I}(x) \setminus \{i\}} c_j(x) |\langle \gamma^j(x), n^i(x) \rangle|. \tag{3.1.21}$$

If additionally Assumption 3.1.12 and

vi) $\{X^n\}$ converges in distribution to a d-dimensional Brownian motion with drift μ, covariance matrix Γ and initial distribution ν.

vii) For each (weak) limit point $\mathcal{Z} = (W, X, Y)$ of $\{Z^n\}_{n=1}^{\infty}$, the compensated process given by $\{X(t) - X(0) - \mu t, \mathcal{F}_t, t \geq 0\}$ is a martingale.

all hold,

then,

$$W^{(n)} \Rightarrow W \quad \text{as } n \to \infty,$$

where W is an SRBM associated with the data $(G, \mu, \Gamma, \{\gamma^i, i \in \mathcal{I}\}, \nu)$.

Remark 3.1.15. *Conditions a) and b) are equivalent by Kang and Williams [51, Lemma 3.1 on p.750].*

We now proceed with our proof of the price convergence. The domain of admissible prices is clearly a convex polyhedron (see Definition 3.1.13), since $\overline{G}$ is defined in terms of $I = 2$, $\beta = (0,0)'$, the inward unit normal vectors $n^1 = (1,0)$ and $n^2 = \frac{1}{\sqrt{2}}(-1,1)$ as for all $(x,y) \in \overline{G}$:

$$\Big\langle n^1, (x,y) \Big\rangle = \Big\langle (1,0), (x,y) \Big\rangle = x \geq 0$$

and

$$\Big\langle n^2, (x,y) \Big\rangle = \Big\langle \frac{1}{\sqrt{2}}(-1,1), (x,y) \Big\rangle = -\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \frac{1}{\sqrt{2}}(y - x) \geq 0.$$

We proceed to show condition a) (3.1.20) of Theorem 3.1.14 and have since $n^1 = \gamma^1$ and $n^2 = \gamma^2$ that $\langle n^1, \gamma^1 \rangle = \langle n^2, \gamma^2 \rangle = 1$ and $|\langle n^1, \gamma^2 \rangle| = |\langle n^2, \gamma^1 \rangle| = \frac{1}{\sqrt{2}}$:

$$b_1 + b_2 = 1, \quad b_1 \geq a + \frac{b_2}{\sqrt{2}}, \quad b_2 \geq a + \frac{b_1}{\sqrt{2}}.$$

The system of equations has e.g. the solution $b_1 = b_2 = \frac{1}{2}$ and $a = \frac{\sqrt{2}-1}{2\sqrt{2}}$ which shows that condition a) (3.1.20) holds.

To show the scaling limit of the price process, we make the following decomposition

$$\hat{P}^{(n)}(t) := X^{(n)}(t) + Y^{(n)}(t), \tag{3.1.22}$$

where $X^{(n)}(t)$ denotes the pure non-reflected zero-intelligence price process and $Y^{(n)}(t)$ the reflection process i.e.

$$X^{(n)}(t) := \left(X_B^{(n)}(t), X_A^{(n)}(t) \right)', \tag{3.1.23}$$

where

$$\begin{cases} X_B^{(n)}(t) := \hat{B}_0 + \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \Delta x \left(-\mathbb{1}_k^A + \mathbb{1}_k^B \right) \\ X_A^{(n)}(t) := \hat{A}_0 + \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \Delta x \left(\mathbb{1}_k^E - \mathbb{1}_k^F \right) \end{cases} \tag{3.1.24}$$

and

$$Y^{(n)}(t) := \left(Y_B^{(n)}(t), Y_A^{(n)}(t)\right)', \tag{3.1.25}$$

where

$$\begin{cases} Y_B^{(n)}(t) := \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \Delta x \left(\mathbb{1}_{\left\{\hat{B}_k^{(n)} = \Delta x^{(n)}\right\}} - \mathbb{1}_{\left\{\hat{A}_k^{(n)} - \hat{B}_k^{(n)} = \Delta x^{(n)}\right\}} \right) \\ Y_A^{(n)}(t) := \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \Delta x \mathbb{1}_{\left\{\hat{A}_k^{(n)} - \hat{B}_k^{(n)} = \Delta x^{(n)}\right\}}. \end{cases} \tag{3.1.26}$$

Aided by the decomposition (3.1.22) above we have the following result for the zero-intelligence part $X^{(n)}$ of the price process.

Lemma 3.1.16 (Convergence of the zero-intelligence price process)**.** *Under Assumption 3.1.7, the zero-intelligence price process $X^{(n)}$ defined in (3.1.23)-(3.1.24) converges weakly in the Skorokhod space $D([0,\infty),\mathbb{R}^2)$ to a two-dimensional Brownian motion starting in $(\hat{B}_0, \hat{A}_0)'$:*

$$X^{(n)}(t) \Rightarrow X(t) \quad \text{as } n \to \infty$$

where the covariance matrix of the Brownian motion X is given by

$$\Gamma := \Delta x^2 \begin{pmatrix} p^A + p^B & 0 \\ 0 & p^E + p^F \end{pmatrix},$$

Δx is the price tick in the original model and p^I is the probability of event I.

Proof. The process $X^{(n)}$ is càdlàg and starts in $(\hat{B}_0, \hat{A}_0)'$ a.s. The expectation of the increments is given by

$$\mathbb{E}\left[\Delta x \left(-\mathbb{1}_k^A + \mathbb{1}_k^B, \mathbb{1}_k^E - \mathbb{1}_k^F\right)'\right] = \Delta x \left(-p^A + p^B, p^E - p^F\right)' = \mathbf{0}$$

by Assumption (3.1.8). For the expectation of the increment products we have, since we assume identically categorically distributed event indicators for all k:

$$\begin{aligned} \Gamma &= \mathbb{E}\left[\Delta x \left(-\mathbb{1}_k^A + \mathbb{1}_k^B, \mathbb{1}_k^E - \mathbb{1}_k^F\right)' \cdot \Delta x \left(-\mathbb{1}_k^A + \mathbb{1}_k^B, \mathbb{1}_k^E - \mathbb{1}_k^F\right)\right] \\ &= \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix}, \end{aligned}$$

where

$$\Gamma_{11} = \mathbb{E}\left[\Delta x^2\left((\mathbb{1}_k^A)^2 - 2\mathbb{1}_k^A\mathbb{1}_k^B + (\mathbb{1}_k^B)^2\right)\right] = \Delta x^2\mathbb{E}\left[\mathbb{1}_k^A + \mathbb{1}_k^B\right] = \Delta x^2\left(p^A + p^B\right)$$
$$\Gamma_{12} = \Gamma_{21} = \mathbb{E}\left[\Delta x^2\left(-\mathbb{1}_k^A\mathbb{1}_k^E + \mathbb{1}_k^A\mathbb{1}_k^F + \mathbb{1}_k^B\mathbb{1}_k^E - \mathbb{1}_k^B\mathbb{1}_k^F\right)\right] = \mathbf{0}$$
$$\Gamma_{22} = \mathbb{E}\left[\Delta x^2\left((\mathbb{1}_k^E)^2 - 2\mathbb{1}_k^E\mathbb{1}_k^F + (\mathbb{1}_k^F)^2\right)\right] = \Delta x^2\mathbb{E}\left[\mathbb{1}_k^E + \mathbb{1}_k^F\right] = \Delta x^2\left(p^E + p^F\right).$$

Since the indicator variables are assumed to be independent in event time k, it follows from e.g. Davidson [21, Corollary 29.19 on p. 494] that $X^{(n)}$ converges weakly to the Brownian motion X, starting in $(\hat{B}_0, \hat{A}_0)'$ with covariance matrix Γ, as $n \to \infty$. □

We now study the reflection process $Y^{(n)}$ of (3.1.22) defined in (3.1.25)-(3.1.26) and show that the càdlàg price process $\hat{P}^{(n)}$ converges to the continuous reflection process $\hat{W}$, by showing that the conditions i)-vi) of Assumption 3.1.12 hold. For this matter, we set $\delta^n := \sqrt{2}\Delta x^{(n)} - \epsilon^n$, where $\epsilon^n := \frac{\Delta x(\sqrt{2}-1)}{n^\gamma}$ with $\gamma > 1$. We notice that clearly $\hat{P} \in U_{\delta^n}(G)$ almost surely. Specifically for the process α^n in condition i) of Assumption 3.1.12, we set $\alpha^n := 0$, as in our case we want the prices to live exactly on the set of price ticks in $G^{(n)} = \left\{x, y \in \{\Delta x^{(n)}, 2\Delta x^{(n)}, \ldots\} \mid x \leq y + \Delta x^{(n)}\right\}$. Also, we have that

$$\hat{P}^{(n)}(t) = X^{(n)}(t) + \int_{(0,t]} (\gamma^1)' \mathrm{d}Y_1^{(n)}(s) + \int_{(0,t]} (\gamma^2)' \mathrm{d}Y_2^{(n)}(s),$$

where $\gamma^1 = (1, 0)$ and $\gamma^2 = \frac{1}{\sqrt{2}}(-1, 1)$.

We proceed and consider the decomposition

$$Y^{(n)}(t) := \widetilde{Y}^n(t) + \beta^n(t)$$

where $\widetilde{Y}^n(0) := 0$ and $\widetilde{\beta}^n(0); = 0$ since we assume that best bid and ask price start in the interior of the price process (i.e. there is no reflection at time $t = 0$) by Assumption 3.1.7. Further,

$$\widetilde{Y}_i^n(t) = \int_{(0,t]} \mathbb{1}_{\{\hat{P}^{(n)}(s) \in U_{\delta^n}(\partial G_i \cap \partial G)\}} \mathrm{d}\widetilde{Y}_i^n(s) \quad \text{for } i = 1, 2$$

$$\beta_i^n(t) = \int_{(0,t]} \mathbb{1}_{\{\hat{P}^{(n)}(s) \in U_{\delta^n}(\partial G_i \cap \partial G)\}} \mathrm{d}\beta_i^n(s) \quad \text{for } i = 1, 2. \tag{3.1.27}$$

so $\Delta\widetilde{Y}_1^n(t) = \Delta\widetilde{Y}_2^n(t) = \delta^n$ and $\Delta\beta_1^n(t) = \Delta\beta_2^n(t) = \epsilon^n$ which means that $\widetilde{Y}^n$ and β^n are nondecreasing processes.

Remark 3.1.17. *For technical reasons, it is necessary to define δ^n as above and to introduce the compensating process β^n. This follows from the condition in Assumption*

3.1.12 that the reflecting vectors γ^i must be unit vectors. At the spread, the reflection vector is $\frac{1}{\sqrt{2}}(-1,1)$ and in order for the process to reflect on a price tick multiple, we must multiply this quantity with the reflection jump of $\sqrt{2}\Delta x^{(n)}$. However, at the same time we want to allow for a one tick spread. Thus, we need $\Delta\widetilde{Y}_2^n(t) \leq \delta^n < \sqrt{2}\Delta x^{(n)}$. If we would allow $\delta^n \leq \sqrt{2}\Delta x^{(n)}$, then we would have

$$\mathbb{1}_{\{\hat{P}^{(n)}(s)\in U_{\sqrt{2}\Delta x^{(n)}}(\partial G_2\cap\partial G)\}} = 1$$

if $\hat{P}^{(n)}(s) \in \{(x,y) \in G^{(n)} \mid x = y + 2\Delta x^{(n)}\}$ and one would have jumps already for a two tick spread by a simple geometric argument, see Figure 3.2.

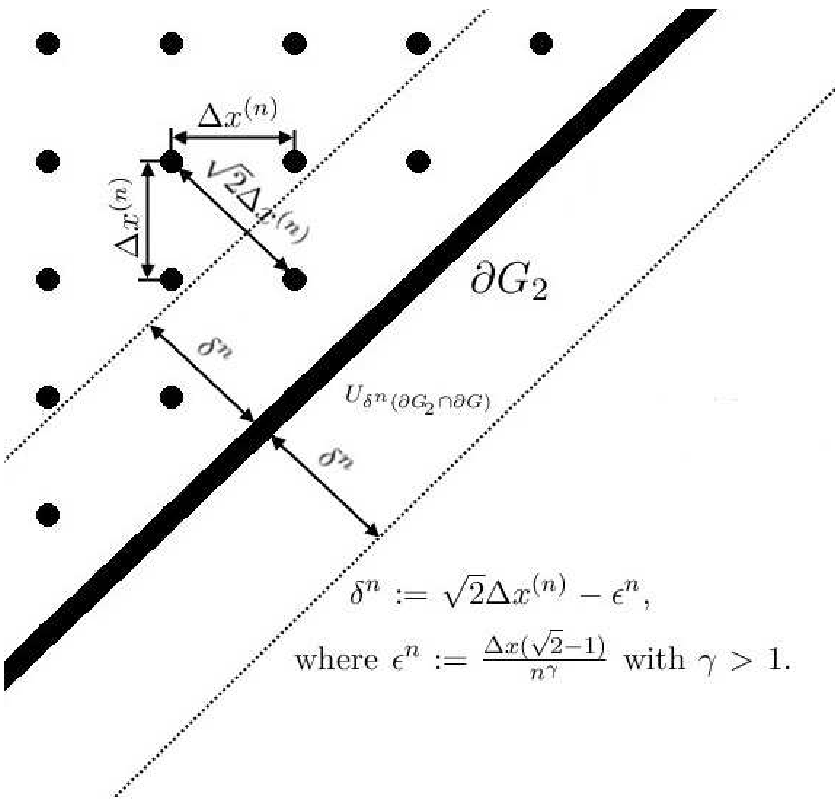

Figure 3.2: Close up of the boundary ∂G_2 and illustration of the geometric argument used in Remark 3.1.17.

We now show that β^n is of locally bounded variation (see Definition 5.3.1 in the Appendix) and that its finite variation process converges to 0 almost surely as $n \to \infty$. β^n jumps by ϵ^n every time the random walk $\hat{P}^{(n)}$ is close to the boundary. By the scaling of $\hat{P}^{(n)}$, the number of jumps is $\lfloor nt \rfloor + 1$ in the time interval $[0,t]$ and this is consequently an upper bound for the number of times β^n moves and we have

$$\mathcal{V}(\beta^n)(t) = \|\beta^n(0)\| + \sup\left\{\sum_{i=1}^{l} |\beta^n(t_i) - \beta^n(t_{i-1})|_2 \mid 0 = t_0 < t_1 < \cdots < t_l = t,\ l \geq 0\right\}$$

$$
\begin{aligned}
&\leq 0 + \sum_{i=1}^{\lfloor nt \rfloor} \|\beta^n(t_i) - \beta^n(t_{i-1})\| \\
&\leq nt \cdot \sqrt{(\epsilon^n)^2 + (\epsilon^n)^2} \\
&= nt\sqrt{2}\frac{\Delta x(\sqrt{2}-1)}{n^\gamma} = \mathcal{O}\left(\frac{1}{n^{\gamma-1}}\right) = o(1) \quad \text{a.s.}
\end{aligned}
$$

as we have $\gamma > 1$ by construction and this implies the convergence in probability. By Lemma 3.1.16, $X^{(n)}$ converges in distribution to a Brownian motion and thus $\{X^{(n)}\}$ is C-tight.

What remains to show is condition vii) of Theorem 3.1.14. By the assumptions of the event indicator variables and the scaling of $X^{(n)}$, we have that $\{X^{(n)}(t) - \hat{X}^{(n)}(0)\}_{n\geq 1}$ is uniformly integrable, the drift of the limiting Brownian motion is zero and $\{X^{(n)}\}_{n\geq 1}$ is a martingale w.r.t. to the natural filtration generated by the event indicator variables. By Kang and Williams [51, Proposition 4.1 on p.766] this implies the condition vii) which we wanted to prove.

Convergence of the relative densities

To prove the convergence of the relative densities, we consider the non-normalized fluctuations $v_b^{(n)}(x,t) = f(Z_b^{(n)}(x,t))$ $[v_s^{(n)}(x,t) = f(Z_s^{(n)}(x,t))]$, where $f(x) := |x|$ and

$$
Z_b^{(n)}(x,t) := \frac{1}{\sigma_b n}\sum_{k=0}^{\lfloor nt \rfloor}\sum_{j=0}^{\lfloor nx \rfloor} \xi_{b,k}^j \qquad \left[Z_s^{(n)}(x,t) := \frac{1}{\sigma_s n}\sum_{k=0}^{\lfloor nt \rfloor}\sum_{j=0}^{\lfloor nx \rfloor} \xi_{s,k}^j\right].
$$

As $n \to \infty$, more and more price levels i.e blocks form in the time-price interval $[0,1] \times [0,1]$. By Poghosyan and Rœlly [64, Theorem 3 on p. 241], the weak convergence

$$
Z_b^{(n)}(x,t) \Rightarrow W_b^{(n)}(x,t) \qquad \left[Z_s^{(n)}(x,t) \Rightarrow W_s(x,t)\right] \quad \text{as } n \to \infty,
$$

where $W_b^{(n)}(x,t)$ $[W_s(x,t)]$ is a 2-parameter Brownian motion, follows. Since $|\cdot|$ is a continuous function the standing buy [sell] volume density converges to a reflected 2-parameter Brownian motion by the continuous mapping principle, see e.g. Billingsley [12, Theorem 2.7 on p.21].

3.1.4 Adding a Drift to the Price Process

As it is possible to step in front of limit orders if the spread is larger than one tick and spreads tend to be small over time, it is natural to let the price process drift toward the spread boundary. The events that involve stepping in front i.e. the willingness to pay for execution priority are $\mathbf{B}$ $[\mathbf{F}]$:= {buy [sell] limit order placed in the spread}. We assume that there are two kinds of these events, which we call sub-events of the I. and II. kind, as a result of zero-intelligence and impatience, respectively:

$\mathbf{B}_I$ $[\mathbf{F}_I]$:= {buy [sell] limit order placed in the spread (zero-intelligence)}

and

$\mathbf{B}_{II}$ $[\mathbf{F}_{II}]$:= {buy [sell] limit order placed in the spread (impatience)}.

Thus, $\mathbf{B} = \mathbf{B}_I \cup \mathbf{B}_{II}$ $[\mathbf{F} = \mathbf{F}_I \cup \mathbf{F}_{II}]$.

We consider the price process on the sequence of probability spaces $(\Omega^{(n)}, \mathcal{F}^{(n)}, \mathbb{P}^{(n)})$ and define

$$\hat{P}^{\#,(n)}(t) := X^{\#,(n)}(t) + Y^{(n)}(t), \tag{3.1.28}$$

where we let the reflecting process $Y^{(n)}$ be just as above in (3.1.25) and the price process away from the boundary be defined as

$$X^{\#,(n)}(t) := \left(X_B^{\#,(n)}(t), X_A^{\#,(n)}(t)\right)', \tag{3.1.29}$$

where

$$\begin{cases} X_B^{\#,(n)}(t) := \hat{B}_0^{(n)} + \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \Delta x \left(-\mathbb{1}_k^{(n),A} + \mathbb{1}_k^{(n),B_I}\right) + \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \Delta x \mathbb{1}_k^{(n),B_{II}} \\ X_A^{\#,(n)}(t) := \hat{A}_0^{(n)} + \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \Delta x \left(\mathbb{1}_k^{(n),E} - \mathbb{1}_k^{(n),F_I}\right) - \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \Delta x \mathbb{1}_k^{(n),F_{II}}. \end{cases} \tag{3.1.30}$$

To define the dynamics completely, we need further assumptions.

Assumption 3.1.18. *We assume that for all n:*

the event indicators $\mathbb{1}_k^{(n),A}, \dots, \mathbb{1}_k^{(n),F_{II}}$ *are independent and identically categorically distributed with the event probabilities*

$$p^{(n),I} := \mathbb{P}\left(\mathbb{1}_k^{(n),I} = 1\right) \in (0,1)$$

such that

$$p^{(n),A} + p^{(n),B_I} + p^{(n),B_{II}} + p^{(n),E} + p^{(n),F_I} + p^{(n),F_{II}} = 1,$$
$$p^{(n),A} = p^{(n),B_I} \quad \textit{and} \quad p^{(n),E} = p^{(n),F_I}.$$

The scaling of the impatient events of type II are given by

$$p^{(n),B_{II}} = \frac{p^{B_{II}}}{\sqrt{n}} \quad \textit{and} \quad p^{(n),F_{II}} = \frac{p^{F_{II}}}{\sqrt{n}}.$$

This assumption means that the price process (3.1.29)-(3.1.30) is a two-dimensional biased random walk and we have the following limit result for this process.

Lemma 3.1.19 (Convergence of the price process with drift)**.** *Under Assumptions 3.1.7 and 3.1.18, the price process $X^{\#,(n)}$ defined in (3.1.29)-(3.1.30) converges weakly in the Skorokhod space $D([0,\infty),\mathbb{R}^2)$ to a two-dimensional Brownian motion $X^{\#}$ starting in $(\hat{B}_0,\hat{A}_0)'$:*

$$X^{\#,(n)}(t) \Rightarrow X^{\#}(t) \quad \text{as } n \to \infty$$

with drift vector $\mu^{\#} = \Delta x \begin{pmatrix} p^{B_{II}} \\ -p^{F_{II}} \end{pmatrix}$ *and covariance matrix*

$$\Gamma^{\#} = \Delta x^2 \begin{pmatrix} p^{\#,A} + p^{\#,B_I} & 0 \\ 0 & p^{\#,E} + p^{\#,F_I} \end{pmatrix},$$

where Δx is the price tick, $p^{B_{II}}$ and $p^{F_{II}}$ are the probabilities of the events B_{II} and F_{II} in the original model, respectively, and $p^{\#,I}$ is the limiting probability of the event $I \in \{A, B_I, E, F_I\}$ i.e.

$$p^{\#,I} := \lim_{n\to\infty} p^{(n),I}.$$

Proof. The process $X^{\#,(n)}$ is càdlàg and starts in $(\hat{B}_0^{(n)},\hat{A}_0^{(n)})$ a.s. by construction.

We consider the first and second sums of 3.1.30, separately and show that the buy [sell] drift process converge to an affine drift function

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor nt\rfloor} \Delta x \mathbb{1}_k^{(n),B_{II}} \to \Delta x p^{B_{II}} t, \quad \text{in probability as } n\to\infty$$
$$\left[\frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor nt\rfloor} \Delta x \mathbb{1}_k^{(n),F_{II}} \to -\Delta x p^{F_{II}} t, \quad \text{in probability as } n\to\infty\right].$$

To see this, we show convergence of the characteristic function, which proves convergence in distribution and thus convergence in probability as the limit is deterministic. For this matter, we denote $\check{X}^{(n)}(t) := \frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor nt\rfloor} \Delta x \mathbb{1}_k^{(n),B_{II}}$ and notice that for each n, this is a sum of independent identically distributed indicator variables. Thus, using Theorem 5.1.10 in the Appendix, we have for the characteristic function recalling the scaling for the price tick $\Delta x^{(n)} = \frac{\Delta x}{\sqrt{n}}$, the probability $p^{(n),B_{II}} = \frac{p^{B_{II}}}{\sqrt{n}}$ and using the Taylor expansion of the exponential function that

$$
\begin{aligned}
\varphi_{\check{X}^{(n)}(t)}(z) &= \left(\varphi_{\Delta x^{(n)}\mathbb{1}_k^{(n),B_{II}}}(z)\right)^{\lfloor nt \rfloor} \\
&= \left(\mathbb{E}\left[e^{i\Delta x^{(n)}\mathbb{1}_k^{(n),B_{II}}z}\right]\right)^{\lfloor nt \rfloor} \\
&= \left((1-p^{(n),B_{II}})e^0 + p^{(n),B_{II}}e^{i\Delta x^{(n)}z}\right)^{\lfloor nt \rfloor} \\
&= \left(1-p^{(n),B_{II}} + p^{(n),B_{II}}\{1+i\Delta x^{(n)}z + \mathcal{O}\left((\Delta x^{(n)}z)^2\right)\}\right)^{\lfloor nt \rfloor} \\
&= \left(1+p^{(n),B_{II}}i\Delta x^{(n)}z + \mathcal{O}\left(p^{(n),B_{II}}(\Delta x^{(n)}z)^2\right)\right)^{\lfloor nt \rfloor} \\
&= \left(1+\frac{p^{B_{II}}}{\sqrt{n}}i\frac{\Delta x}{\sqrt{(n)}}z + \mathcal{O}\left(\frac{p^{B_{II}}}{\sqrt{n}}(\frac{\Delta x}{\sqrt{(n)}}z)^2\right)\right)^{\lfloor nt \rfloor} \\
&= \left(1+\frac{i\Delta x p^{B_{II}}z}{n} + \mathcal{O}\left(\frac{z^2}{n^{3/2}}\right)\right)^{\lfloor nt \rfloor} \\
&= \left(1+\frac{1}{n}\left(i\Delta x p^{B_{II}}z + \mathcal{O}\left(\frac{1}{n^{1/2}}\right)\right)\right)^{\lfloor nt \rfloor} \to e^{i\Delta x p^{B_{II}}tz} \quad \text{as } n\to\infty
\end{aligned}
$$

and this proves the convergence of the finite-dimensional distributions of $\check{X}^{(n)}$ to the drift process (for the buy side, the sell side may be shown equivalently) using Lévy's Continuity Theorem 5.1.11 of the Appendix. The drift process μt is continuous and for each t, $\{\check{X}^{(n)} : 1 \leq n < \infty\}$ is a uniformly integrable martingale. This implies the weak convergence of $\check{X}^{(n)}$ to the buy side of the drift process, see Proposition 5.1.12 of the Appendix. Thus, we have shown the convergence of the drift part and continue with showing the convergence of the zero-intelligence part.

The expectation of the increments is given by

$$
\begin{aligned}
\mathbb{E}\left[\Delta x\left(-\mathbb{1}_k^{(n),A}+\mathbb{1}_k^{(n),B_I}, \mathbb{1}_k^{(n),E}-\mathbb{1}_k^{(n),F}\right)'\right] &= \Delta x\left(-p^{(n),A}+p^{(n),B_I}, p^{(n),E}-p^{(n),F}\right)' \\
&= \mathbf{0} \quad \text{for all } n \geq 1
\end{aligned}
$$

by Assumption (3.1.8). For the expectation of the increment products we have, since we assume identically categorically distributed event indicators for all k:

$$
\begin{aligned}
\Gamma^{(n)} &= \mathbb{E}\left[\Delta x\begin{pmatrix} -\mathbb{1}_k^{(n),A}+\mathbb{1}_k^{(n),B_I} \\ \\ \mathbb{1}_k^{(n),E}-\mathbb{1}_k^{(n),F_I}\end{pmatrix} \cdot \Delta x\left(-\mathbb{1}_k^{(n),A}+\mathbb{1}_k^{(n),B_I}, \mathbb{1}_k^{(n),E}-\mathbb{1}_k^{(n),F_I}\right)\right] \\
&= \begin{pmatrix} \Gamma_{11}^{(n)} & \Gamma_{12}^{(n)} \\ \Gamma_{21}^{(n)} & \Gamma_{22}^{(n)} \end{pmatrix},
\end{aligned}
$$

where

$$\begin{aligned}
\Gamma_{11}^{(n)} &= \mathbb{E}\left[\Delta x^2 \left((\mathbb{1}_k^{(n),A})^2 - 2\mathbb{1}_k^{(n),A}\mathbb{1}_k^{(n),B_I} + (\mathbb{1}_k^{(n),B_I})^2\right)\right] = \Delta x^2 \mathbb{E}\left[\mathbb{1}_k^{(n),A} + \mathbb{1}_k^{(n),B_I}\right] \\
&= \Delta x^2 \left(p^{(n),A} + p^{(n),B_I}\right) \\
\Gamma_{12}^{(n)} &= \Gamma_{21}^{(n)} = \mathbb{E}\left[\Delta x^2 \left(-\mathbb{1}_k^{(n),A}\mathbb{1}_k^{(n),E} + \mathbb{1}_k^{(n),A}\mathbb{1}_k^{(n),F_I} + \mathbb{1}_k^{(n),B_I}\mathbb{1}_k^{(n),E} - \mathbb{1}_k^{(n),B_I}\mathbb{1}_k^{(n),F_I}\right)\right] \\
&= \mathbf{0} \\
\Gamma_{22}^{(n)} &= \mathbb{E}\left[\Delta x^2 \left((\mathbb{1}_k^{(n),E})^2 - 2\mathbb{1}_k^{(n),E}\mathbb{1}_k^{(n),F_I} + (\mathbb{1}_k^{(n),F_I})^2\right)\right] = \Delta x^2 \mathbb{E}\left[\mathbb{1}_k^{(n),E} + \mathbb{1}_k^{(n),F_I}\right] \\
&= \Delta x^2 \left(p^{(n),E} + p^{(n),F_I}\right).
\end{aligned}$$

As the indicator variables are assumed to be independent in event time k, it follows from e.g. Davidson [21, Theorem 29.18 on p. 492] that $X^{\#,(n)}$ converges weakly to the Brownian motion $X^{\#}$, starting in $(\hat{B}_0, \hat{A}_0)'$ with drift vector $\mu^{\#}$ and covariance matrix $\Gamma^{\#} = \lim_{n\to\infty} \Gamma^{(n)}$, as $n \to \infty$. □

We thus state the following extension to the FCLT 3.1.10.

Theorem 3.1.20 (Functional Central Limit Theorem for a simple order book model with drift)**.** *If Assumptions 3.1.7 and 3.1.18 hold for the model with price dynamics defined in (3.1.28)-(3.1.30). Then, we have that*

$$\hat{P}^{\#,(n)}(t) = \left(\hat{B}^{\#,(n)}(t), \hat{A}^{\#,(n)}(t)\right)' \Rightarrow \hat{W}^{\#}(t) \quad \text{as } n \to \infty \tag{3.1.31}$$

where $\hat{W}^{\#}(t)$ is a two-dimensional semi-martingale reflecting Brownian motion associated with the data $(G, \mu^{\#}, \Gamma^{\#}, \{\gamma_i, i \in \mathcal{I}\}, \nu)$, where the set of admissible prices is given by

$$G = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le y\right\}, \tag{3.1.32}$$

the drift vector is $\mu = \Delta x \begin{pmatrix} p^{B_{II}} \\ -p^{F_{II}} \end{pmatrix}$, *the covariance matrix is given by*

$$\Gamma^{\#} = \Delta x^2 \begin{pmatrix} p^{\#,A} + p^{\#,B_I} & 0 \\ 0 & p^{\#,E} + p^{\#,F_I} \end{pmatrix},$$

and the boundaries of G in (3.1.32)

$$\partial G_1 := \left\{(x, y) \in \mathbb{R}_+^2 \mid x = 0\right\} \quad \text{and} \quad \partial G_2 := \left\{(x, y) \in \mathbb{R}_+^2 \mid y = x\right\}$$

have constant reflection vectors

$$\gamma^1 := (1, 0) \quad \text{and} \quad \gamma^2 := \frac{1}{\sqrt{2}}(-1, 1).$$

respectively. The initial distribution ν *is singular i.e.*

$$\nu\Big(\{(\hat{B}(0), \hat{A}(0))' = (\hat{B}_0, \hat{A}_0)' \in G \setminus \partial G\}\Big) = 1. \tag{3.1.33}$$

Let the sequence of order book models be zero intelligence models, then the standing buy [sell] volume density

$$\hat{v}_b^{(n)}(x,t) \Rightarrow |W_b(x,t)| \quad \left[\hat{v}_s^{(n)}(x,t) \Rightarrow |W_s(x,t)|\right] \quad \text{as } n \to \infty, \tag{3.1.34}$$

where $W_b(x,t)$ $[W_s(x,t)]$ *is a 2-parameter Brownian motion and* $(x,t) \in [0,1] \times [0,1]$.

Proof. The convergence of the price process follows by Lemma 3.1.19 and the analogous method of proof as used for Theorem 3.1.10. As we assume that the relative volume densities are ergodic with respect to price shifts, the weak convergence of the volume densities follows analogously as in the proof of Theorem 3.1.10. □

In analogy to Corollary 3.1.11, we get an approximation in absolute price coordinates close to the spread in the drift-case as well.

Corollary 3.1.21 (Time evolution of standing limit order volume close to the spread)**.** *For the limit model, the observable standing limit order density at time* t *is given by*

$$w(p,t) := \left|W_b\left(\hat{B}^{\#}(t) - p, t\right)\right| \mathbb{1}_{[\hat{B}^{\#}(t)-1, \hat{B}^{\#}(t))}(p) + \left|W_s\left(p - \hat{A}^{\#}(t), t\right)\right| \mathbb{1}_{[\hat{A}^{\#}(t), \hat{A}^{\#}(t)+1)}(p),$$

where $p \in [\hat{B}^{\#}(t) - 1, \hat{A}^{\#}(t) + 1]$ *denotes the price of the security,*

the best bid [ask] price $\hat{B}^{\#}$ $[\hat{A}^{\#}]$ *is the* x $[y]$ *component of the SRBM with drift* $\hat{W}^{\#}$ *in (3.1.31) of Theorem 3.1.20,*

W_b $[W_s]$ *is a 2-parameter Brownian motion for* $(\hat{B}^{\#}(t) - p) \in [0,1]$ $\left[(p - \hat{A}^{\#}(t)) \in [0,1]\right]$ *and equal to 0 for* $(\hat{B}^{\#}(t) - p) \notin [0,1]$ $\left[(p - \hat{A}^{\#}(t)) \notin [0,1]\right]$.

Below in Figures 3.3-3.6, we have plotted the prices of INTC and AAPL and compare these with with simulated paths for two different choices of parameters of the discrete price process suggested above. The limiting price process and sensible extensions are discussed in the conclusion at the end of the chapter.

3.2 Dominating Price Changes

In this section, we prove a Functional Law of Large Numbers/Invariance Principle for a simple case of the order book dynamics given in Chapter 2. We will see that the limiting dynamics for the best bid and ask prices are given as Brownian motions and the relative buy and sell volume densities as the solutions of parabolic, second order SPDE:s, see Pardoux [61, p.35].

Assumption 3.2.1 (SPDE scaling)**.** *We assume that:*

- *The tick scalings are given by*

$$\Delta x^{(n)} := \frac{\Delta x}{\sqrt{n}}, \qquad \Delta v^{(n)} := \frac{\Delta v}{\sqrt{n}}, \qquad \Delta t^{(n)} := \frac{\Delta t}{n}.$$

- *Inter arrivals of market orders and limit orders within the spread are equidistant, deterministic and always one time tick $\Delta t^{(n)}$.*

- *The fluctuations of the order book i.e. limit order placement/cancelation (the impact of the events C, D, G, H) is negligible and not taken into account in the dynamics.*

- *The best bid [ask] price tick multiples are integer valued IID random variables $\{\xi_k\}_{k\in\mathbb{N}}$ where k is the event time, $\mathbb{E}[\xi_k]=0$ and $\mathbb{E}[\xi_k^2]=1$. We assume a constant spread. Thus, the best bid [ask] price changes for the prices occur simultaneously and the evolution in continuous time is given by the càdlàg process*

$$\overline{B}^{(n)}(t) = B_0 + \frac{\Delta x}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \xi_k \quad \left[\overline{A}^{(n)}(t) = A_0 + \frac{\Delta x}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \xi_k \right]. \tag{3.2.1}$$

- *For the initial relative buy [sell] volume density $v_{b,0}^{(n)}$ $[v_{s,0}^{(n)}]$ the regularity of Assumption 2.1.6 holds and the dynamics in continuous time are given by*

$$\overline{v}_b^{(n)}(x,t) = v_{b,0}^{(n)}\left(x + \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \xi_k\right) \quad \left[\overline{v}_s^{(n)}(x,t) = v_{s,0}^{(n)}\left(x + \frac{1}{\sqrt{n}} \sum_{k=0}^{\lfloor nt \rfloor} \xi_k\right)\right]. \tag{3.2.2}$$

The main result of this section is as follows.

Theorem 3.2.2. *Suppose that Assumption 3.2.1 holds, then the best bid and ask prices*

(3.2.1) converge weakly

$$\overline{B}^{(n)}(t) \Rightarrow B^*(t) = B_0 + \Delta x W(t), \quad \overline{A}^{(n)}(t) \Rightarrow A^*(t) = A_0 + \Delta x W(t) \tag{3.2.3}$$

in $D([0,\infty),\mathbb{R})$ *as* $n \to \infty$ *and* W *is standard Brownian motion.*

The relative buy and sell volume densities converge weakly

$$\overline{v}_b^{(n)}(x,t) \Rightarrow v_b^*(x,t) = v_{b,0}\Big(x + \Delta x W(t)\Big), \quad \Big[\overline{v}_s^{(n)}(x,t) \Rightarrow v_s^*(x,t) = v_{s,0}\Big(x + \Delta x W(t)\Big)\Big], \tag{3.2.4}$$

in the pointwise sense on $\mathbb{R} \times D([0,\infty),\mathbb{R})$ *as* $n \to \infty$*, where the limiting relative buy and sell volume densities* v_b^* *and* v_s^* *solve the SPDE:s*

$$\begin{cases} \frac{\partial v_b^*(x,t)}{\partial t} &= \frac{\partial^2 v_b^*(x,t)}{\partial x^2} + \Delta x \frac{\partial v_b^*(x,t)}{\partial x}\frac{\mathrm{d}W(t)}{\mathrm{d}t} \\ v_b^*(x,t) &= v_{b,0}(x) \end{cases} \tag{3.2.5}$$

and

$$\begin{cases} \frac{\partial v_s^*(x,t)}{\partial t} &= \frac{\partial^2 v_s^*(x,t)}{\partial x^2} + \Delta x \frac{\partial v_s^*(x,t)}{\partial x}\frac{\mathrm{d}W(t)}{\mathrm{d}t} \\ v_s^*(x,t) &= v_{s,0}(x) \end{cases} \tag{3.2.6}$$

respectively.

Proof. The weak convergence of the best bid and ask prices (3.2.3) follows by the Donsker Invariance Principle on $D([0,\infty),\mathbb{R})$, for a proof see e.g. the book by Ethier and Kurtz [28, p.278].

One has that the convergence of the relative volume densities (3.2.4) holds, since by the mean value theorem the a.e. continuous initial relative densities $v_{b,0}^{(n)} \to v_{b,0}$, $v_{s,0}^{(n)} \to v_{s,0}$ pointwise as $n \to \infty$. Thus, by the generalized Continuous Mapping Theorem in Whitt [76, p.68] the pointwise weak convergence of the densities (3.2.2) follows by the weak convergence of the prices (3.2.3). Finally, a simple application of Îto's formula to the limiting relative densities v_b^* and v_s^* yields the SPDE:s (3.2.5) and (3.2.6). □

Again, we get an approximation in absolute price coordinates as a corollary:

Corollary 3.2.3 (Time evolution of standing limit order volume)**.** *For the limiting model, the observable standing limit order density at time* t *is given by*

$$w^*(p,t) := v_b^*\left(\overline{B}(t) - p, t\right) \mathbb{1}_{[0,\overline{B}(t))}(p) + v_s^*\left(p - \overline{A}(t), t\right) \mathbb{1}_{[\overline{A}(t),\infty)}(p),$$

where p *denotes the price of the security, the best bid and ask prices* $\overline{B}, \overline{A}$ *are given by*

(3.2.3) and the relative buy and sell volume densities v_b^*, v_s^* solve (3.2.5) and (3.2.6) of Theorem 3.2.2 respectively.

3.3 Conclusion and Outlook

Our results in the first section of the chapter yield a continuous approximation for the bid and ask price processes and the volumes close to the spread, in the sense of a functional central limit theorem. In the limiting model, the prices are given by an SRBM and the volumes close to the spread as a reflected 2-parameter Brownian motion. In Figures 3.7 and 3.8, we have simulated a path of the price process over the unit time interval $[0, 1]$ with the choice of parameters used for the simulation of the discrete model used in Figure 3.4 and 3.6. The mean-reverting spread property of the simulated path resembles that of the data.

It is important to point out that in this chapter, the inter arrival times are deterministic and equidistant. However, one can extend our modeling to include recursively defined random inter arrival times. In principle, this could be done as in Chapter 2 (see (2.1.2)) using state and time separation (Proposition 4.2.6) and the time change theorem (Theorem 4.2.7) which also holds for weak convergence. Thus, if the scaled state and time processes, each converge weakly, their composition also converges weakly. In this sense, our framework provides us with a wealth of modeling possibilities and enables us to move away from the Brownian paths of the price and volume processes to include jumps, suggested by many authors e.g. Cont and Tankov [18] and Hudson and Mandelbrot [47]. To give a simple example, assume that the limiting state process is that of Theorem 3.1.10 and the time process converges to an increasing Lévy process. Away from the boundary, the limiting price process would be a subordinated Brownian motion and the limiting volumes the moduli of subordinated 2-parameter Brownian motions. There exists a rich literature on this subject, see e.g. Cont and Tankov [18, p.114-120] concerning subordinated Brownian motion and Dobrushin [24] concerning subordinated Gaussian random fields. A closer study of the properties of these processes, especially involving the reflection processes, would be very interesting but is beyond the scope of this thesis.

Also, we do not model the coupling of prices and volumes explicitly but give mathematical assumptions under which our FCLT holds. It would be interesting to endogenize the price process further, i.e. that volume placements effect the price dynamics. It seems that the independence assumptions on the event indicators for the discrete price process (Assumptions 3.1.7) may be weakened. There exist many general results in this direction e.g. for near epoch dependent random walks that converge to Brownian motion, see Davidson [21, Theorem 29.19 on p.494]. Concerning the weakening of the volume assumptions, the recent paper by El Machkouri et. al. [26], in which a FCLT for stationary fields converging to generalized white noise (see Definition 5.3.2 in the

Appendix) was proven, could be useful.

In Section 3.2 we derived an SPDE for a simple case of the model introduced in Chapter 2. We only modeled the price shifts, and in general the prices could converge to more general processes as long as they are differentiable in the sense of Itô. This means that our approach could be used to yield more general SPDE:s and together with our detailed study of the dynamics and scaling rates in Chapter 2, we are quite optimistic that it should be possible, while at the same time challenging, to derive SPDE:s when the limit order arrivals and cancelations are dynamic in nature. An important step would be to find a finite difference approximation of such an SPDE, which just as in Chapter 2 needs to be coherent with the discrete order book dynamics. Roth studied finite difference schemes for first order hyperbolic SPDE:s in his thesis [68], but the scheme would correspond to limit order placement/cancelation at each price simultaneously. Numerics for SPDE:s is a very active field of research. It is quite possible that general methods such as in Jentzen and Kloeden [48] could be used to derive an appropriate approximating scheme, which converges to the solution of an SPDE, and is coherent with the scaled dynamics of the discrete model considered in Chapter 2.

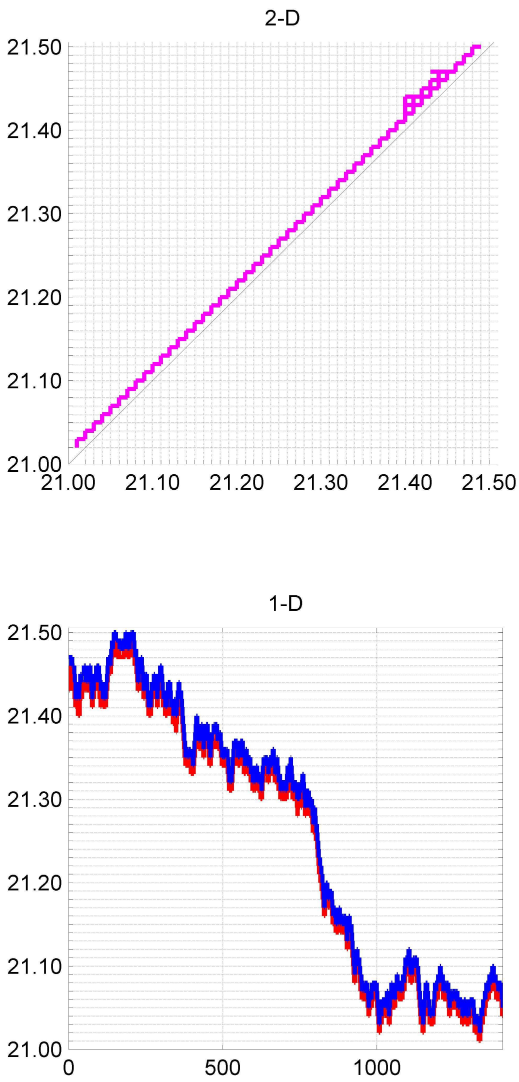

Figure 3.3: The bid and ask prices of INTC from the 4th of April 2013 in active order event time. There were 1408 active orders placed in INTC at NASDAQ on that day.

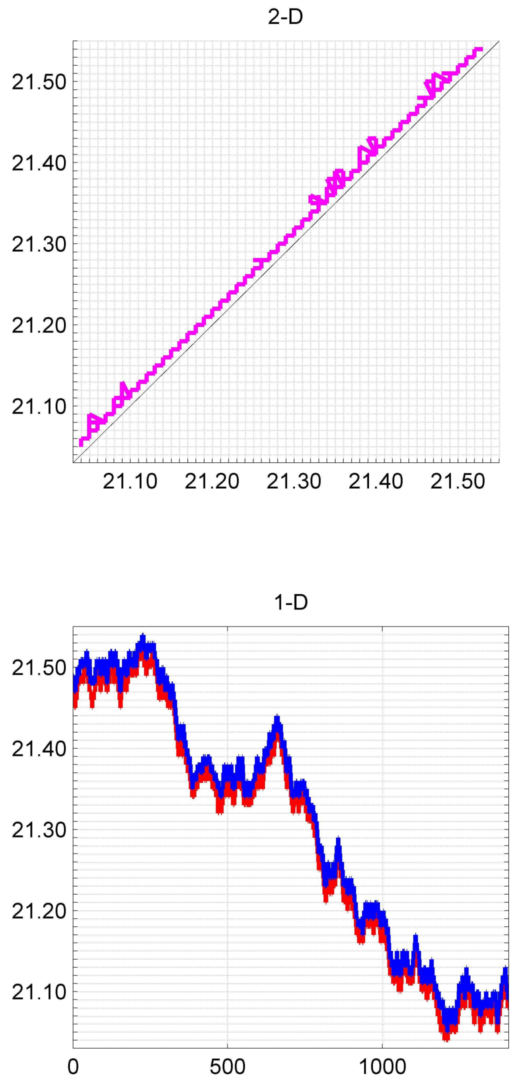

Figure 3.4: Simulation of 1408 events of the prices in the discrete model, defined in (3.1.28)-(3.1.30). The initial bid and ask price are 21.47 and 21.49 respectively, the original price tick is $\Delta x = 0.01$ and the drift probabilities are $p_{B_{II}} = p_{F_{II}} = 0.49$.

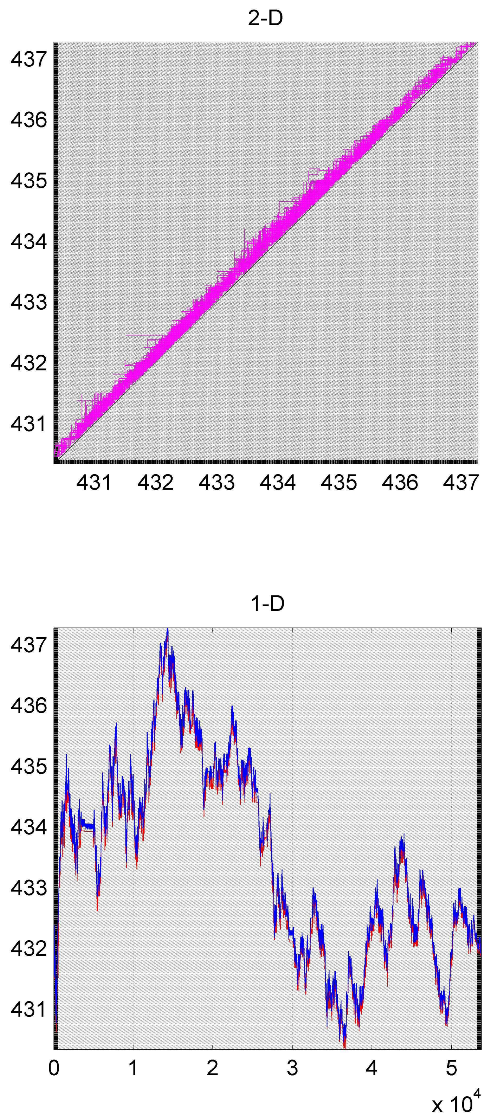

Figure 3.5: The bid and ask prices of AAPL from the 4th of April 2013 in active order event time. There were 53745 active orders placed in AAPL at NASDAQ on that day.

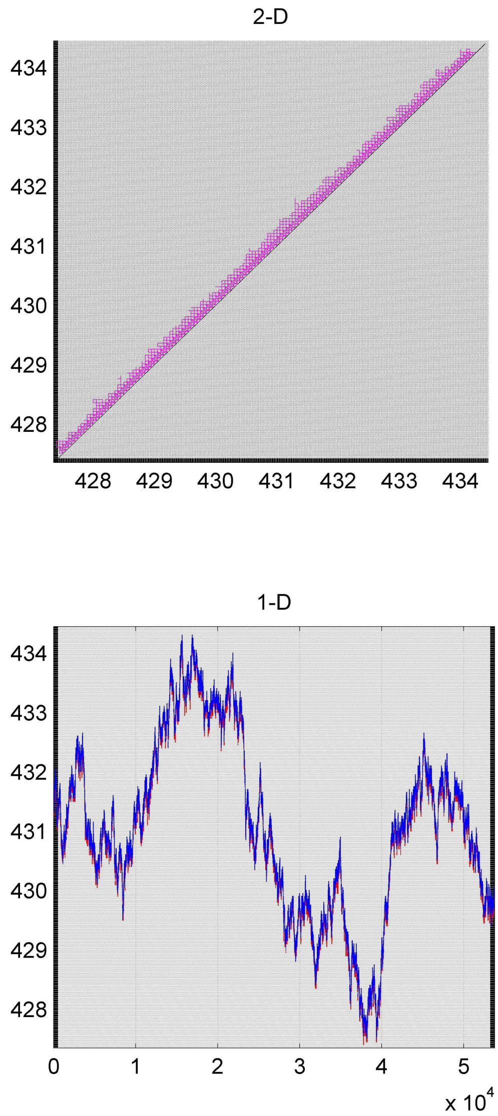

Figure 3.6: Simulation of 53745 events of the prices in the discrete model, defined in (3.1.28)-(3.1.30). The initial bid and ask price are 431.30 and 430.81 respectively, the price jumps are $5 \cdot \Delta x = 0.05$ and the drift probabilities are $p_{B_{II}} = p_{F_{II}} = 0.3$.

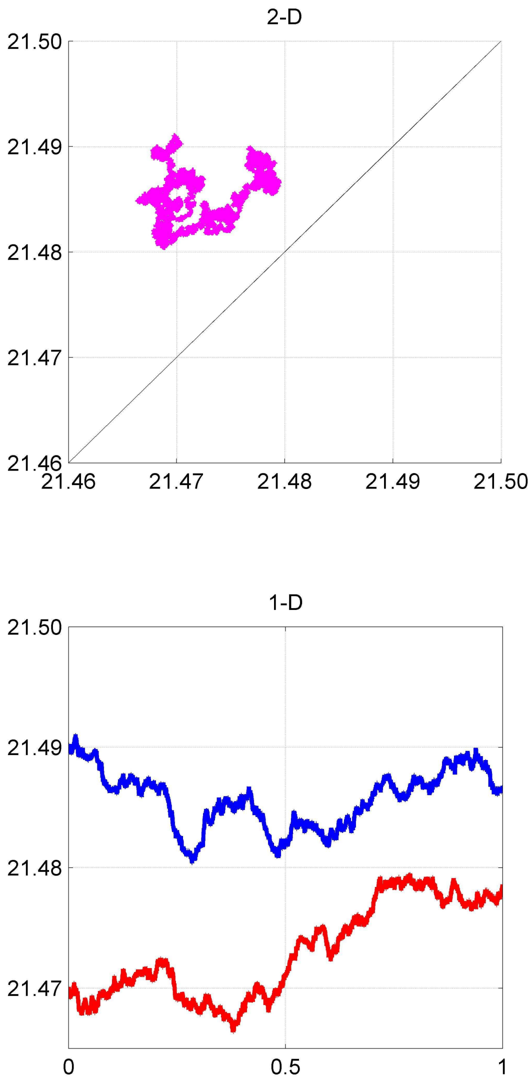

Figure 3.7: Simulation of the limiting SRBM over the unit active order time interval $[0, 1]$. The initial bid and ask price are 21.47 and 21.49 respectively, the original price tick is $\Delta x = 0.01$ and the drift probabilities are $p_{B_{II}} = p_{F_{II}} = 0.49$.

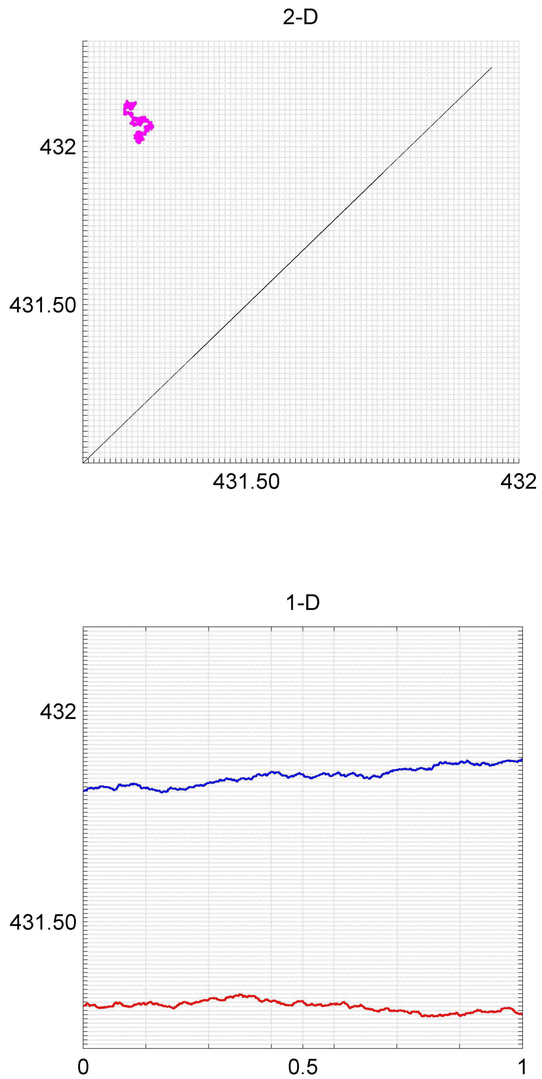

Figure 3.8: Simulation of the limiting SRBM over the unit active order time interval $[0,1]$. The initial bid and ask price are 431.30 and 430.81 respectively, the price jumps are $5 \cdot \Delta x = 0.05$ and the drift probabilities are $p_{B_{II}} = p_{F_{II}} = 0.3$.

4 Limit Theorems in Banach spaces

In the first section of this chapter, we define a mathematical framework for processes of the kind defined in Chapters 2 and 3, that may take values in real separable Banach spaces which are p-uniformly smooth for $p \in (1, 2]$. The geometric restriction of uniform smoothness is needed for the Strong Law of Large Numbers by Hoffman-Jorgensen and Pisier (Theorem 4.2.4). We then proceed to prove an averaging principle in the spirit of Anisimov [6, Theorem 3.1 p.107] for this abstract setting.

The second section is devoted to reviewing sufficient conditions for a diffusion approximation (i.e. a functional central limit theorem or invariance principle) to hold, when the noise is state-independent and the state space is a real separable Hilbert space. At the end of the chapter, we conclude our results and give an outlook for extensions.

4.1 Mathematical Framework

We begin by stating our mathematical framework, which generalizes the setting used in Chapters 2 and 3.

Let $(E, \|\cdot\|)$ be a real separable Banach space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space such that $\mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}$, where

$$\mathcal{F}_{-1} := \sigma(s_0) \quad \text{for a finite variance random variable } s_0 \in E \tag{4.1.1}$$

and the

$$\mathcal{F}_k := \sigma\left(s_0, \{(\Xi_i(\alpha), \Phi_i(\alpha)), \alpha \in E\}_{i=0}^k\right), \quad k \geq 0, \tag{4.1.2}$$

are jointly conditional independent families of random variables with values in $E \times [0, \infty)$. Given $\alpha \in E$, we assume that the random variables $\Xi_k(\alpha)$ and $\Phi_k(\alpha)$ are identically distributed for all $k \geq 0$.

The event times $\tau_k^{(n)}$ are defined recursively as

$$\tau_0^{(n)} := 0, \quad \tau_{k+1}^{(n)} := \tau_k^{(n)} + \mathcal{C}_k^{(n)}(S_k^{(n)}), \quad \text{for all } k \geq 0. \tag{4.1.3}$$

The state dynamics are defined as

$$S_0^{(n)} := s_0^{(n)}, \quad S_{k+1}^{(n)} := S_k^{(n)} + \mathcal{D}_k^{(n)}(S_k^{(n)}), \tag{4.1.4}$$

where $s_0^{(n)} := f(n, s_0) \in E$. We assume that the random operators $\mathcal{C}_k^{(n)} : E \to \mathbb{R}_+$ and $\mathcal{D}_k^{(n)} : E \to E$ take the form

$$\mathcal{C}_k^{(n)}(\alpha) = d^{(n)}(\Phi_k(\alpha), \alpha) \cdot \Delta t^{(n)} \tag{4.1.5}$$

$$\mathcal{D}_k^{(n)}(\alpha) = a^{(n)}(\Xi_k(\alpha), \alpha) \cdot \Delta t^{(n)} \tag{4.1.6}$$

where $d^{(n)}$ and $a^{(n)}$ are measurable functions, for all n. Thus, for all n the elements of the sequence of processes $\{S^{(n)}\}_{n\geq 1}$ are defined on the same probability space. The constant $\Delta t^{(n)}$ denotes the time increment and we assume that $\Delta t^{(n)} \to 0$, as $n \to \infty$. This means that the scaling of state and time are of the same order $\Delta t^{(n)}$.

When observing the process in continuous time, we have

$$S^{(n)}(t) := S_k^{(n)} \quad \text{as} \quad t \in [\tau_k^{(n)}, \tau_{k+1}^{(n)}), \quad t \geq 0. \tag{4.1.7}$$

It follows that the paths of the stochastic process $S^{(n)}(t)$ are càdlàg, i.e. are right continuous and have left limits and we write $S^{(n)}(t)(\omega) \in D([0,\infty), E)$.

Our convergence results will be for $t \in [0, T]$, and thus we have

$$S^{(n)}(\cdot) : (\Omega, \mathcal{F}, \mathbb{P}) \to \Big(D([0,T], E), \mathcal{B}(D([0,T], E))\Big).$$

When the limiting process is continuous in the time parameter (as we shall see that the process is) we may endow $D([0,T], E)$ with the uniform topology, with respect to the norm (see Billingsley [12, p.124]):

$$\|f\|_T := \sup_{t\in[0,T]} \|f(t)\|, \quad \text{for } f \in D([0,T], E). \tag{4.1.8}$$

4.2 An Averaging Principle in Banach Space

To prove our Averaging Principle, the notion of martingale difference sequences on Banach spaces will be key.

Definition 4.2.1 (Martingale difference sequence on Banach space). *Let $\{X_k\}$ be a finite or infinite sequence of integrable E-valued random variables, where E is a Banach* *space. If*

i) *there exist* σ*-algebras* $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$*, such that* X_k *is* $\mathcal{F}_k$*-measurable, for all* k *and*

ii) $\mathbb{E}[X_{k+1}|\mathcal{F}_k] = 0$*, for all* k

the sequence $\{X_k\}$ *is said to be a martingale difference sequence.*

The Strong Law of Large Numbers for martingale difference sequences by Hoffman-Jorgensen and Pisier (Theorem 4.2.4) holds for Banach spaces with a favorable geometry. These Banach spaces are called p-uniformly smooth, where $p \in (1, 2]$ and include the spaces L^p, l^p and the Sobolev spaces W^p_m, see Woyczyński [78].

Definition 4.2.2 (Modulus of smoothness)**.** *Let* E *be a Banach space. The modulus of smoothness of* E *is defined as*

$$\rho_E(\tau) = \sup\left\{\|\frac{x+y}{2}\| + \|\frac{x-y}{2}\| - 1 : \|x\| = 1, \|y\| = \tau\right\}.$$

E *is said to be* p*-uniformly smooth,* $p \in (1, 2]$ *if*

$$\rho_E(\tau) \leq C\tau^p$$

for some constant C*.*

Remark 4.2.3. *It holds that all Hilbert spaces are 2-uniformly smooth by the parallelogram identity. The spaces* C*,* l^1 *and* L^1 *are not uniformly smooth.*

Theorem 4.2.4 (Strong Law of Large Numbers for martingale difference sequences on Banach space, Hoffman-Jorgensen and Pisier [43, Theorem 2.2 on p. 591])**.** *Let* $p \in (1, 2]$ *and* $\{X_k\}_{k\geq 1}$ *be a martingale difference sequence taking values in the Banach space* E *such that*

$$\sum_{k=1}^{\infty} \frac{\mathbb{E}\left[\|X_k\|^p\right]}{k^p} < \infty. \tag{4.2.1}$$

Then the Strong Law of Large Numbers

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to 0 \quad a.s.$$

holds if and only if E *is isomorphic to some uniformly* p*-smooth space.*

The main result of this section is the following.

Theorem 4.2.5 (Averaging Principle)**.** *Let $(E, \|\cdot\|)$ be a separable Banach space that is p-uniformly smooth with $p \in (1,2]$. If*

i) $\Delta t^{(n)} = \frac{1}{n}$.

ii) The sequences of the expected values of the time- and state-increment functions $d^{(n)}$ in (4.1.5) and $a^{(n)}$ in (4.1.6) converge pointwise as $n \to \infty$:

$$m^{(n)}(\alpha) := \mathbb{E}[d^{(n)}(\Phi_1(\alpha), \alpha)] \to m^*(\alpha), \quad b^{(n)}(\alpha) := \mathbb{E}[a^{(n)}(\Xi_1(\alpha), \alpha)] \to b^*(\alpha), \tag{4.2.2}$$

for all $\alpha \in E$ where the functions $m^{(n)}, m^* : E \to \mathbb{R}_+$ and $b^{(n)}, b^* : E \to E$ are globally Lipschitz continuous with Lipschitz constants $L_{m^{(n)}}, L_{m^*}, L_{b^{(n)}}$ and L_{b^*}, respectively.

iii)

$$\frac{\mathbb{E}\left[\|\psi_k^{(n)}(\alpha)\|^p\right]}{k^p} = \mathcal{O}\left(\frac{1}{k^{1+\epsilon}}\right) \quad \textit{and} \quad \frac{\mathbb{E}\left[\|\varphi_k^{(n)}(\alpha)\|^p\right]}{k^p} = \mathcal{O}\left(\frac{1}{k^{1+\epsilon}}\right), \tag{4.2.3}$$

uniformly in n, $\alpha \in E$ and some $\epsilon > 0$, where

$$\psi_k^{(n)}(\alpha) := d^{(n)}(\Phi_k(\alpha), \alpha) - m^{(n)}(\alpha) \quad \textit{and} \quad \varphi_k^{(n)}(\alpha) := a^{(n)}(\Xi_k(\alpha), \alpha) - b^{(n)}(\alpha).$$

iv)

$$S_0^{(n)} \to s_0 \quad \textit{a.s. as } n \to \infty. \tag{4.2.4}$$

Then, we have that

$$\sup_{t \in [0,T]} \left\| S^{(n)}(t) - s(t) \right\| \to 0 \quad \textit{a.s. as } n \to \infty, \tag{4.2.5}$$

where $s(t)$ is the unique solution to the ODE

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = \frac{b^*(s(t))}{m^*(s(t))}, \qquad s(0) = s_0 \in E. \tag{4.2.6}$$

The upper boundary of the time interval T is any positive number such that $y(+\infty) > T$ a.s. where

$$y(t) = \int_0^t m^*(\eta(u))\mathrm{d}u, \tag{4.2.7}$$

and $\eta(t)$ is the unique solution of the ODE

$$\frac{\mathrm{d}\eta(u)}{\mathrm{d}u} = b^*(\eta(u)), \qquad \eta(0) = s_0 \in E. \tag{4.2.8}$$

Proof of Theorem 4.2.5

A series of auxiliary results will be used in proving the Averaging Principle. The first proposition is used to split up the process $S^{(n)}$ in two simpler processes.

Proposition 4.2.6 (State and time separation, Anisimov [6, Proof of Theorem 3.1 on p.108-109]). *The process $S^{(n)}$ can be expressed as the composition of a random state process $\eta^{(n)}$ and a random time process $\mu^{(n)}$ that change at deterministic points in time, i.e.*

$$S^{(n)}(t) = \eta^{(n)}\left(\mu^{(n)}(t) - \frac{1}{n}\right), \text{ for all } t > 0$$

where

$$\mu^{(n)}(t) := \inf\{u : u > 0, y^{(n)}(u) > t\}, \tag{4.2.9}$$

$$y^{(n)}(u) := \tau_k^{(n)} \quad \text{and} \quad \eta^{(n)}(u) := S_k^{(n)}, \text{ for} \quad u \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \Leftrightarrow nu \in [k, k+1), \quad u \geq 0. \tag{4.2.10}$$

Proof. Following Anisimov [6, p.108] we define

$$\nu^{(n)}(t) := \min\{k : k > 0, \tau_{k+1}^{(n)} > t\},$$

$$\mu^{(n)}(t) := \inf\{u : u > 0, y^{(n)}(u) > t\}.$$

It follows that $y^{(n)}(\frac{\nu^{(n)}(t)}{n}) \leq t < y^{(n)}(\frac{\nu^{(n)}(t)}{n} + 1)$ and $\mu^{(n)}(t) = \frac{\nu^{(n)}(t)+1}{n}$. To see this, we have that $\nu^{(n)}(t) := \min\{k \in \mathbb{N}, \tau_{k+1}^{(n)} > t\}$, so it holds by definition that $\nu^{(n)}(t)$ is the smallest natural number such that $\tau_{\nu^{(n)}(t)+1}^{(n)} > t$ and thus we must have $\tau_{\nu^{(n)}(t)}^{(n)} \leq t$. Therefore,

$$\begin{aligned} y^{(n)}\left(\frac{\nu^{(n)}(t)+1}{n}\right) &= \tau_k^{(n)} \quad \text{for } \nu^{(n)}(t) + 1 \in [k, k+1) \\ &> t \end{aligned}$$

and

$$\begin{aligned} y^{(n)}\left(\frac{\nu^{(n)}(t)}{n}\right) &= y_k^{(n)} \\ &= \tau_k^{(n)} \quad \text{for } \nu^{(n)}(t) \in [k, k+1) \\ &\leq t. \end{aligned}$$

Further $\mu^{(n)}(t) = \frac{\nu^{(n)}(t)+1}{n}$, since we just showed that $y^{(n)}(\frac{\nu^{(n)}(t)+1}{n}) > t$ and the choice of $\nu^{(n)}(t)$ is optimal by the definition of ν and y.

Now, since

$$S^{(n)}(t) = S_k^{(n)} \quad \text{for} \quad t \in [\tau_k^{(n)}, \tau_{k+1}^{(n)}),$$

we have $S^{(n)}(t) = S^{(n)}_{\nu^{(n)}(t)}$, for all $t > 0$. Hence, we may write

$$S^{(n)}(t) = S^{(n)}_{\nu^{(n)}(t)} = \eta^{(n)}\left(\frac{\nu^{(n)}(t)}{n}\right) = \eta^{(n)}\left(\mu^{(n)}(t) - \frac{1}{n}\right), \quad \text{for all } t > 0.$$

□

The following time change theorem states that the almost sure convergence of the state and time process above, implies the almost sure convergence of their composition on a compact time interval.

Theorem 4.2.7 (Time Change Theorem, Billingsley [12, p.151]). *Let $(E, \|\cdot\|)$ be a separable Banach space. If*

i) the sequence of stochastic processes $\{X^{(n)}(t)\}_{n\geq 1} \in D([0,T], E)$ converges a.s. to $X(t) \in C([0,T], E)$,

ii) the sequence of non-decreasing stochastic processes $\{\Phi^{(n)}(t)\}_{n\geq 1} \in D([0,T], [0,T])$ converges a.s. to $\{\Phi(t)\} \in C([0,T], [0,T])$.

Then

$$X^{(n)} \circ \Phi^{(n)} \to X \circ \Phi \quad \textit{a.s as } n \to \infty. \tag{4.2.11}$$

Proof. We work out the details of the proof, outlined in Billingsley [12, p.151], using the fact that E is a separable and complete metric space under the metric $\|x - y\|$, for $x, y \in E$.

The mapping $\Phi := (X, \Phi) := X \circ \Phi$ is in $C([0,T], E)$ at (X, Φ), for $X \in C([0,T], E)$, $\Phi \in C([0,T], [0,T])$.

Elements $x^{(n)}$ in $D([0,T], E)$ converge to a limit x in the Skorokhod topology if and only if there exist functions

$$\lambda^{(n)} \in \Lambda := \{\text{strictly increasing cont. mappings from } [0,T] \text{ to } [0,T]\}$$

such that $\lim_n x^{(n)}(\lambda^{(n)}t) = x(t)$ and $\lim_n \lambda^{(n)}t = t$ uniformly in $t \in [0,T]$. (*)

By the above, we may choose $\lambda^{(n)}$ such that $\sup_{t\in[0,T]} |\Phi^{(n)}(\lambda^{(n)}t) - \Phi(t)| \to 0$ and

$\sup_{t\in[0,T]} |\lambda^{(n)}t - t| \to 0$. We write

$$\begin{aligned}\|X^{(n)}\Phi^{(n)}t - X^{(n)}\Phi^{(n)}\lambda^{(n)}t\| &\le \|X^{(n)}\Phi^{(n)}t - X\Phi^{(n)}t\| + \|X\Phi^{(n)}t - X^{(n)}\Phi^{(n)}\lambda^{(n)}t\| \\ &\le \sup_{u\in[0,T]} \|X^{(n)}(u) - X(u)\| + \sup_{s,u\in\mathcal{T}^{(n)}} \|X(s) - X(u)\| \\ &\to 0 \quad \text{as } n \to \infty,\end{aligned}$$

since X is continuous on $[0,T]$ (Skorokhod convergence implies uniform convergence at continuity points) and $\mathcal{T}^{(n)} := \sup|\Phi^{(n)}(t) - \Phi\lambda^{(n)}(t)| \to 0$ as $n \to \infty$, by the above. Using (*) once again, the claim (4.2.11) follows. □

The discrete version of the Gronwall lemma will be used repeatedly to show the convergence of the state dependent processes.

Lemma 4.2.8 (Discrete Gronwall lemma, see e.g. Holte [44, Lemma 1 on p.1] or Elaydi [27, Lemma 4.32 on p.220])**.** *Let $\{y_m\}_{m\ge0}$, $\{f_m\}_{m\ge0}$ and $\{g_m\}_{m\ge0}$ be nonnegative sequences. If*

$$y_m \le f_m + \sum_{k=0}^{m-1} g_k y_k$$

then

$$y_m \le f_m + \sum_{k=0}^{m-1} f_k g_k e^{\sum_{j=k+1}^{m-1} g_j}.$$

We are ready now, to prove the main result of the section.

Proof of Theorem 4.2.5:
By Proposition 4.2.6 $S^{(n)}(t)$ can be expressed as the composition of a state process $\eta^{(n)}(t)$ and a time process $\mu^{(n)}(t) - \frac{1}{n}$. If we show that the state and time processes converge individually, we may apply Theorem 4.2.7 to conclude that their composition $S^{(n)}(t)$ converges as well.

We claim that the state process $\eta^{(n)}$ converges, i.e.

$$\sup_{t\in[0,T]} \|\eta^{(n)}(t) - \eta(t)\| \to 0 \quad \text{a.s. as } n \to \infty, \tag{4.2.12}$$

where η will be shown to be the unique solution of (4.2.8).

Using the defining recurrent equations (4.1.3)-(4.1.4), we write the processes $\eta^{(n)}$ and

$y^{(n)}$, defined in Proposition 4.2.6, equivalently as

$$\eta^{(n)}(t)=\eta^{(n)}_{\lfloor nt\rfloor}=S_0^{(n)}+\sum_{k=0}^{\lfloor nt\rfloor} b^{(n)}(\eta^{(n)}(t_k^{(n)}))\Delta t^{(n)}+\sum_{k=0}^{\lfloor nt\rfloor}\varphi_k^{(n)}(\eta^{(n)}(t_k^{(n)}))\Delta t^{(n)} \tag{4.2.13}$$

$$y^{(n)}(t)=y^{(n)}_{\lfloor nt\rfloor}=S_0^{(n)}+\sum_{k=0}^{\lfloor nt\rfloor} m^{(n)}(\eta^{(n)}(t_k^{(n)}))\Delta t^{(n)}+\sum_{k=0}^{\lfloor nt\rfloor}\psi_k^{(n)}(\eta^{(n)}(t_k^{(n)}))\Delta t^{(n)},$$

with

$$\begin{aligned} m^{(n)}(\alpha)&:=\mathbb{E}[d_1^{(n)}(\alpha)], \quad \psi_k^{(n)}(\alpha):=d_k^{(n)}(\alpha)-\mathbb{E}[d_1^{(n)}(\alpha)]\\ b^{(n)}(\alpha)&:=\mathbb{E}[a_1^{(n)}(\alpha)], \quad \varphi_k^{(n)}(\alpha):=a_k^{(n)}(\alpha)-\mathbb{E}[a_1^{(n)}(\alpha)]\end{aligned}$$

for $\alpha\in E$. The sequences $\left\{\psi_k^{(n)}(\eta_k^{(n)})\right\}_{k\geq 0}$ and $\left\{\varphi_k^{(n)}(\eta_k^{(n)})\right\}_{k\geq 0}$ are martingale difference sequences on $\mathbb{R}$ and E with respect to the filtration $\mathcal{F}_k$:

$$\begin{aligned}\mathbb{E}\Big[\psi_k^{(n)}(\eta_k^{(n)})\Big|\mathcal{F}_{k-1}\Big]&=\mathbb{E}\Big[d_k^{(n)}(\eta_k^{(n)})-\mathbb{E}[d_1^{(n)}(\eta_k^{(n)})]\Big|\mathcal{F}_{k-1}\Big]\\ &=\mathbb{E}\Big[d_k^{(n)}(\eta_k^{(n)})\Big|\mathcal{F}_{k-1}\Big]-\mathbb{E}[d_1^{(n)}(\eta_k^{(n)})]\\ &=\mathbb{E}[d_1^{(n)}(\eta_k^{(n)})]-\mathbb{E}[d_1^{(n)}(\eta_k^{(n)})]\\ &=0\end{aligned} \tag{4.2.14}$$

and likewise for $\left\{\varphi_k^{(n)}(\eta_k^{(n)})\right\}_{k\geq 0}$, since we assumed that the stochastic variables that generate $\mathcal{F}_k$ are independent, given the current state.

Also, we define the processes

$$\chi^{(n)}(t):=s_0+\sum_{k=0}^{\lfloor nt\rfloor} b^*(\chi_k^{(n)})\Delta t^{(n)} \tag{4.2.15}$$

$$\zeta^{(n)}(t):=s_0+\sum_{k=0}^{\lfloor nt\rfloor} b^{(n)}(\zeta_k^{(n)})\Delta t^{(n)}.$$

and using this with (4.2.13) we can write

$$\begin{aligned}\|\eta^{(n)}(t)-\eta(t)\|&=\|\eta^{(n)}_{\lfloor nt\rfloor}-\eta(t)\|\\ &\leq\|\eta^{(n)}_{\lfloor nt\rfloor}-\zeta^{(n)}_{\lfloor nt\rfloor}\|+\|\zeta^{(n)}_{\lfloor nt\rfloor}-\chi^{(n)}_{\lfloor nt\rfloor}\|+\|\chi^{(n)}_{\lfloor nt\rfloor}-\eta(t)\|\end{aligned} \tag{4.2.16}$$

and consider the convergence of the three right-hand terms of (4.2.16) separately.

We have that

$$\begin{aligned}
\|\eta^{(n)}_{\lfloor nt\rfloor} - \zeta^{(n)}_{\lfloor nt\rfloor}\| &= \|S^{(n)}_0 + \sum_{k=0}^{\lfloor nt\rfloor} b^{(n)}(\eta^{(n)}_k)\Delta t^{(n)} + \sum_{k=0}^{\lfloor nt\rfloor} \varphi^{(n)}_k(\eta^{(n)}_k)\Delta t^{(n)} \\
&\qquad - s_0 - \sum_{k=0}^{\lfloor nt\rfloor} b^{(n)}(\zeta^{(n)}_k)\Delta t^{(n)}\| \\
&\le \|S^{(n)}_0 - s_0\| + \|\sum_{k=0}^{\lfloor nt\rfloor} \left\{ b^{(n)}(\eta^{(n)}_k) - b^{(n)}(\zeta^{(n)}_k) \right\} \Delta t^{(n)}\| \\
&\qquad + \|\sum_{k=0}^{\lfloor nt\rfloor} \varphi^{(n)}_k(\eta^{(n)}_k)\Delta t^{(n)}\| \\
&\le \|S^{(n)}_0 - s_0\| + \Delta t^{(n)} \sum_{k=0}^{\lfloor nt\rfloor} \|b^{(n)}(\eta^{(n)}_k) - b^{(n)}(\zeta^{(n)}_k)\| \\
&\qquad + \|\Delta t^{(n)} \sum_{k=0}^{\lfloor nt\rfloor} \varphi^{(n)}_k(\eta^{(n)}_k)\| \\
&\le \|S^{(n)}_0 - s_0\| + L_{b^{(n)}} \sum_{k=0}^{\lfloor nt\rfloor} \Delta t^{(n)} \|\eta^{(n)}_k - \zeta^{(n)}_k\| + \|\Delta t^{(n)} \sum_{k=0}^{\lfloor nt\rfloor} \varphi^{(n)}_k(\eta^{(n)}_k)\|
\end{aligned} \tag{4.2.17}$$

using the Lipschitz property of $b^{(n)}$. We have $\Delta t^{(n)} = \mathcal{O}(\frac{1}{n})$, $\left\{\varphi^{(n)}_k(\eta^{(n)}_k)\right\}_{k\ge 0}$ is a martingale difference sequence and iii) is assumed to be true. Thus, convergence of the third term in (4.2.17) follows by the Strong Law of Large Numbers by Hoffman-Jorgensen and Pisier (Theorem 4.2.4). Applying the discrete Gronwall Lemma 4.2.8, we get

$$\|\eta^{(n)}_{\lfloor nt\rfloor} - \zeta^{(n)}_{\lfloor nt\rfloor}\| \le o(1) L_{b^{(n)}} \mathcal{O}(\frac{1}{n}) nT e^{L_{b^{(n)}}\mathcal{O}(1)} + o(1) = o(1) \tag{4.2.18}$$

since $e^{\sum_{j=k+1}^{\lfloor nt\rfloor -1} L_{b^{(n)}}\Delta t^{(n)}} \le e^{L_{b^{(n)}}\mathcal{O}(\frac{1}{n})nT} = e^{L_{b^{(n)}}\mathcal{O}(1)}$.

$$\begin{aligned}
\|\zeta^{(n)}_{\lfloor nt\rfloor} - \chi^{(n)}_{\lfloor nt\rfloor}\| &= \|\sum_{k=0}^{\lfloor nt\rfloor} b^{(n)}(\zeta^{(n)}_k)\Delta t^{(n)} - \sum_{k=0}^{\lfloor nt\rfloor} b^{*}(\chi^{(n)}_k)\Delta t^{(n)}\| \\
&\le \|\sum_{k=0}^{\lfloor nt\rfloor} \left\{ b^{(n)}(\zeta^{(n)}_k) - b^{*}(\zeta^{(n)}_k) \right\} \Delta t^{(n)}\| + \|\sum_{k=0}^{\lfloor nt\rfloor} \left\{ b^{*}(\zeta^{(n)}_k) - b^{*}(\chi^{(n)}_k) \right\} \Delta t^{(n)}\| \\
&\le o(1)\lfloor nt\rfloor \Delta t^{(n)} + L_{b^*} \sum_{k=0}^{\lfloor nt\rfloor} \Delta t^{(n)} \|\zeta^{(n)}_k - \chi^{(n)}_k\|
\end{aligned}$$

$$\leq o(1)nT\mathcal{O}(\frac{1}{n}) + L_{b^*}\sum_{k=0}^{\lfloor nt\rfloor}\Delta t^{(n)}\|\zeta_k^{(n)} - \chi_k^{(n)}\|$$

$$= o(1) + L_{b^*}\sum_{k=0}^{\lfloor nt\rfloor}\Delta t^{(n)}\|\zeta_k^{(n)} - \chi_k^{(n)}\| \tag{4.2.19}$$

since $\Delta t^{(n)} = \mathcal{O}(\frac{1}{n})$, the convergence of $b^{(n)}$ and the Lipschitz property of its limit b^*. Applying Gronwall's discrete Lemma 4.2.8 on (4.2.19), we get

$$\|\zeta_{\lfloor nt\rfloor}^{(n)} - \chi_{\lfloor nt\rfloor}^{(n)}\| \leq o(1) + o(1)L_{b^*}\mathcal{O}(\frac{1}{n})nTe^{L_{b^*}\mathcal{O}(1)} = o(1). \tag{4.2.20}$$

Also, it holds that

$$\|\chi^{(n)}(t) - \eta(t)\| = o(1) \tag{4.2.21}$$

by the convergence of the sum in (4.2.15) to its Riemann Integral on Banach space.

From (4.2.18)-(4.2.21) we conclude that (4.2.12) holds uniformly for all $t \in [0,T]$ since η is continuous. Analogously as was done for η, one can show for the cumulative event time process that

$$\sup_{t\in[0,T]}|y^{(n)}(t) - y(t)| = 0 \quad \text{a.s. as } n \to 0.$$

To show the convergence $S^{(n)}$, i.e. equivalently the convergence of the composition of the processes $\eta^{(n)}$ and $\mu^{(n)}$, we may apply Theorem 4.2.7 on time change, since the processes converge individually, to get (4.2.6) by calculation; since $s(t) = \eta(\mu(t)) = \eta(y^{-1}(t))$, we have by the Chain Rule on Banach spaces (see e.g. Jost [49, Theorem 8.4 on p.105])), along with the fact that η and μ are solutions of ODE:s themselves, that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}s(t) = s'(t) = \eta'(\mu(t))\cdot\mu'(t) &= b^*(\eta(\mu(t)))\cdot(y^{-1}(t))' \\ &= b^*(s(t))\cdot\frac{1}{y'(y^{-1}(t))} \\ &= b^*(s(t))\cdot\frac{1}{m^*(\eta(y^{-1}(t)))} \\ &= \frac{b^*(s(t))}{m^*(s(t))}.\end{aligned} \tag{4.2.22}$$

Since b^* and m^* are Lipschitz continuous and $m^* > 0$, their composition is Lipschitz and thus there exists a unique solution of (4.2.22).

□

4.3 Diffusion Approximation in Hilbert Space

In this section we restrict ourselves to separable real Hilbert spaces H. The reason is that we need the existence of the inner product of H, which we denote by $\langle \cdot, \cdot \rangle$, to define a covariance operator. We denote the norm of H as usual by $\| \cdot \|$ and say that a Hilbert space-valued random variable X has mean or expectation $E[X] \in H$ if $\mathbb{E}[\langle X, h \rangle] = \langle \mathbb{E}[X], h \rangle$ for all $h \in H$. The r:th moment of X will be denoted $\|X\|_r := (\mathbb{E}[\|X\|^r])^{1/r}$. The random variable $\mathcal{N} \in H$ is said to to have a Gaussian distribution if for all $h \in H$, the random variable $\langle h, \mathcal{N} \rangle$ has a Gaussian distribution on $\mathbb{R}$. We write $\{e_l\}$ for a complete orthonormal basis, let $I : H \to H$ denote the identity operator and $L(H, H)$ be the Banach space of bounded linear operators from H to H.

Definition 4.3.1. *An operator $\mathcal{S} : H \to H$ which is symmetric, positive semidefinite, compact and has finite trace is called an $\mathcal{S}$-operator.*

An $L(H, H)$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a covariance operator of V_k given $X_1, \ldots, X_k$ if

$$\langle \mathcal{S}^n y, y \rangle = \mathbb{E}\left[\langle V_k, y \rangle^2 \,|\, X_1, \ldots, X_k \right], \quad y \in H,\, k \geq 0. \tag{4.3.1}$$

By a Brownian motion on the Hilbert space H, we shall mean the following element of $C([0, 1], H)$, where $C([0, 1], H)$ denotes the set of all continuous functions from the time interval $[0, 1]$ taking values in H. This space is a separable Banach space under the norm $\|f\|_\infty := \sup_{t \in [0,1]} \|f(t)\|_H$.

Definition 4.3.2 (Brownian motion on H)**.** *The stochastic process W_H in $C([0, 1], H)$ such that*

i) $W_H(0) = \mathbf{0}$ *a.s.*

ii) Increments on disjoint time intervals are independent.

iii) For all $0 \leq t < t + s \leq 1$ and all $h \in H$, the real-valued random variable $\langle W_H(t + s) - W_H(t), h \rangle$ has a Gaussian distribution on $\mathbb{R}$ with mean $\mathbb{E}[\langle W_H(t + s) - W_H(t), h \rangle] = 0$ and covariance $s\langle \mathcal{S}h, h \rangle$, where $\mathcal{S}$ is a covariance operator that does not depend on $s, t \in [0, 1]$.

is called a Brownian motion on H.

We now consider a second order approximation of the discrete stochastic state process

(4.1.4) for $n = 1$ and consider the deviation of the random state operator and its expectaion at the k:th event:

$$V_k := \mathcal{D}_k^{(1)}\left(S_k^{(1)}\right) - \mathbb{E}\left[\mathcal{D}_k^{(1)}\left(S_k^{(1)}\right)|\mathcal{F}_{k-1}\right]. \tag{4.3.2}$$

The process above is easily seen to be a martingale difference sequence and for such a sequence the following invariance principle holds.

Theorem 4.3.3 (Invariance principle, Walk [75, Theorem 2 on p.142])**.** *Let $\mathcal{S} : E \to E$ be an $\mathcal{S}$-operator and $\mathcal{S}^n$ be the conditional covariance operator. If it holds that*

i)

$$\mathbb{E}\left[\left\|\frac{1}{n}\sum_{k=1}^{n}\mathcal{S}^k - \mathcal{S}\right\|\right] \to 0 \quad \text{as } n \to \infty$$

ii)

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left[\|V_k\|^2\right] \to \mathit{trace}(\mathcal{S}) \quad \text{as } n \to \infty$$

iii) For all $r > 0$:

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left[\|V_k\|^2 \mathbb{1}_{\{\|V_k\|^2 \geq rj \,|\, V_1,\ldots,V_{k-1}\}}\right] \to 0 \quad \text{in probability as } n \to \infty.$$

Then, it holds, that

$$Y_n(t) := \frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor nt\rfloor} V_k + (nt - \lfloor nt\rfloor)\frac{1}{\sqrt{n}}V_{\lfloor nt\rfloor+1} \Rightarrow W_H(t) \quad \text{as } n \to \infty$$

where W_H is a Brownian motion in the Hilbert space H.

From a modeling point of view, it would of course be interesting to be able to prove an invariance principle of a more general character. Thus, we now consider a sequence of the following non-centered variables, where especially n is allowed to vary as well.

$$W_k^{(n)} := \mathcal{D}_k^{(n)}\left(S_k^{(n)}\right). \tag{4.3.3}$$

To formulate the rather general invariance principle, we need some further definitions which provide weaker assumptions than independence of sequential event times. We also introduce the notion of near epoch dependence, which is weaker than the martingale difference assumption used in Chapter 2 and for the averaging principle above. Our discussion follows that of Chen and White [14].

Definition 4.3.4 (α- and ϕ-mixing, near epoch dependent (NED) $L^p(H)$-array)**.** *Let $\{U_k^{(n)}\}_{k\in\mathbb{Z},n\geq 1}$ be an array of Banach space-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define the sequence of σ-algebras $\mathcal{A}_{(n),a}^{(n),b} := \sigma(U_j^{(n)} \mid a \leq j \leq b)$ for all $a \leq b$, where $-\infty \leq a$, $b \leq \infty$ and $n \geq 1$.*

i) The array $\{U_k^{(n)}\}_{k\in\mathbb{Z},n\geq 1}$ is called a strong- or α-mixing array of Banach space-valued random variables if $\lim_{m\to\infty} \alpha_m = 0$, where

$$\alpha_m := \sup_{n\geq 1} \sup_i \sup \left[|\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)| \mid A \in \mathcal{A}_{(n),-\infty}^{(n),i}, B \in \mathcal{A}_{(n),i+m}^{(n),\infty} \right].$$

ii) The array $\{U_k^{(n)}\}_{k\in\mathbb{Z},n\geq 1}$ is called a uniform- or ϕ-mixing array of Banach space-valued random variables if $\lim_{m\to\infty} \phi_m = 0$, where

$$\phi_m := \sup_{n\geq 1} \sup_i \sup \left[|\mathbb{P}(B|A) - \mathbb{P}(B)| \mid A \in \mathcal{A}_{(n),-\infty}^{(n),i}, B \in \mathcal{A}_{(n),i+m}^{(n),\infty} \right].$$

iii) A Hilbert space-valued array $\{W_k^{(n)}\}_{k\geq 0,n\geq 1}$ is called an $L^p(H)$-array near epoch dependent (NED) on the sequence $\{U_k^{(n)}\}_{k\in\mathbb{Z},n\geq 1}\}$ if $\|W_k^{(n)}\|_p < \infty$ for all $k \geq 0, n \geq 1$ and there exist sequences of constants $\{d_k^{(n)} \geq 0\}_{k\geq 0,n\geq 1}$ and $\{\mu_m \geq 0\}_{m\geq 0}$ with μ_m decreasing to zero as $m \to \infty$ such that

$$\|W_k^{(n)} - \mathbb{E}\left[W_k^{(n)} \mid \mathcal{A}_{(n),k+m}^{(n),k-m} \right] \|_p \leq \mu_m d_k^{(n)}. \tag{4.3.4}$$

A practical way to measure the dependence of mixing random variables, is to quantify the rate of convergence of the mixing coefficients α_m and ϕ_m defined above.

Definition 4.3.5 (Mixing size)**.** *We say that a sequence is α-mixing [ϕ-mixing] of size $-\rho_0$ if $\alpha_m = \mathcal{O}(m^{-\rho})$ [$\phi_m = \mathcal{O}(m^{-\rho})$] for some $\rho > \rho_0$.*

Another generalization is given for the scaling i.e. the partitioning of the event time on the interval $[0, 1]$. Whereas in Theorem 4.3.3 and in Chapter 3, we summed over $0 \leq k \leq \lfloor nt \rfloor$, the summation below is done over $0 \leq k \leq a^{(n)}$, where $\{a^{(n)}\}_{n\geq 1}$ is a sequence of nondecreasing, right continuous functions $a^{(n)} : [0, 1] \to \mathbb{N}$ such that $a^{(n)}(0) = 0$ and $a^{(n)}(t) - a^{(n)}(s) \to \infty$ as $n \to \infty$ for all $0 \leq s < t \leq 1$. We consider the process

$$X^{(n)}(t) := \sum_{k=0}^{a^{(n)}(t)} W_k^{(n)}, \quad \text{if } a^{(n)}(t) > a^{(n)}(t-) \text{ and otherwise by linear interpolation.}$$

Finally, we are ready to formulate the, admittedly quite technical, conditions of the following invariance principle by Chen and Williams.

Theorem 4.3.6 (Invariance Principle for H-valued NED Processes, see Chen and White [14, Theorem 4.6 on p.271]). *Let $\{W_k^{(n)}\}_{k\geq 0, n\geq 1}$ be a double array of zero mean Hilbert space-valued random variables with $\|W_k^{(n)}\|_r < \infty$ for some $r \geq 2$. If the following conditions all hold:*

a) *i)* *$\{W_k^{(n)}\}_{k\geq 0, n\geq 1}$ is $L_2(H)$-NED on $\{U_k^{(n)}\}_{k\in\mathbb{Z}, n\geq 1}$ with μ_m of size $-\gamma$, $\gamma \in [\frac{1}{2}, 1]$ and $\max_{0\leq k\leq a^{(n)}(1)} d_k^{(n)} = \mathcal{O}\left((a^{(n)}(1))^{\gamma-1}\right)$.*

ii) *$\{U_k^{(n)}\}_{k\in\mathbb{Z}, n\geq 1}$ is a mixing Banach space-valued array of stochastic variables with either α_m of size $-r/(r-2)$, $r > 2$ or ϕ_m of size $-r/2(r+1)$.*

iii)

$$\sup_{l,J} \limsup_{n\to\infty} \frac{1}{l} \sum_{k=J+1}^{J+l} (c_k^{(n)})^2 < \infty.$$

where $c_k^{(n)} := \max\left\{d_k^{(n)}, \|W_k^{(n)}\|_r\right\}$ for all $k \geq 0, n \geq 1$.

b) *When $r = 2$, the sequence $\left\{\left|\frac{W_k^{(n)}}{c_k^{(n)}}\right|^2\right\}_{k\geq 0, n\geq 1}$ is uniformly integrable for each nonzero $h \in H$.*

c) *For all $t \in [0,1]$ and nonzero $h \in H$, $\lim_{n\to\infty} \mathbb{E}\left[\sum_{k=0}^{a^{(n)}(t)} \langle W_k^{(n)}, h\rangle\right]^2 = t\sigma^2(h)$.*

d) $\limsup_{n\to\infty} \mathbb{E}\left[\left|\sum_{k=0}^{a^{(n)}(1)} W_k^{(n)}\right|^2\right] = C < \infty$

e) *There exists a complete orthonormal basis $\{e_l\}$ such that*

$$\limsup_{n\to\infty} \sum_{l>j} \sum_{k=0}^{a^{(n)}(1)} \langle c_k^{(n)}, e_l\rangle^2 \to \infty, \quad \text{as } j \to \infty.$$

Then, it follows that

i) *There exists a nonsingular covariance operator $S \in \mathcal{S}$ such that $(Sh, h) = \sigma^2(h)$ for each $h \in H$, $h \neq 0$.*

ii) *The sequence of functions*

$$X^{(n)} \Rightarrow W_H \quad \text{in } C([0,1], H), \text{ where } W_H \text{ is a Brownian motion on } H.$$

iii) *$\sum_{k=0}^{a^{(n)}(1)} W_k^{(n)} \Rightarrow \mathcal{N}(0, S)$ as $n \to \infty$, where $\mathcal{N}(0, S)$ denotes a normally distributed* *random variable in H with covariance operator S.*

4.4 Conclusion and Outlook

In Section 4.2 we proved an averaging principle for p-uniformly smooth Banach spaces under a simple scaling in the state-dependent case. The limiting model was given as the unique solution of an ODE on the state space and characterized by the limit of the expectations of the state time operators. We used the Strong Law of Large Numbers by Hoffman-Jorgensen and Pisier (Theorem 4.2.4) to prove the Averaging Principle and the main result of Chapter 2. The proofs, especially for the latter result, could be simplified if convergence rates in the SLLN would be available for the state spaces. In other words, it would be very interesting if the Brunk-Prokhorov Strong Law of Large Numbers for real-valued martingale difference arrays, Theorem 5.2.2 could be extended to p-uniformly smooth Banach spaces or at least Hilbert spaces as these conditions in the applications to volume densities would be quite easy to verify (the second moment of the density fluctuations have already been calculated in Chapter 2). There exist some results for these spaces concerning the convergence speed in Marcinkievicz-Zygmund type SLLNs, see Dedecker and Merlevède [22] and it should be possible to use these results to extend the Averaging Principle stated in Theorem 4.2.5 for other scalings and weaker conditions.

Concerning generalized FCLTs, we applied existing results by Walk [75] and Chen and White [14] to give diffusion limits in Section 4.3. More precisely, we provided a second-order approximation of the discrete model by showing weak convergence to Brownian motion on Hilbert spaces in two simple cases. In the more interesting case of state-dependent noise, we expect the limit to be an SDE on Banach space or Hilbert space. For this kind of result it would be useful to study Euler-type schemes for SDEs on abstract spaces see e.g. Schomerus [70]. Since SPDEs may be considered as SDEs on functions spaces in general and Hilbert spaces in particular (see e.g. Prévot and Röckner [66]), this undertaking could also be interesting for deriving SPDEs for limit order book models, see Section 3.2.

5 Appendix

5.1 General Definitions and Results

We recall the following notation.

Definition 5.1.1 (Big oh and little oh, see e.g. Apostol [8, Definition 8.24 on p.192])**.** *Let $f, g : \mathbb{N} \to \mathbb{R}$ be functions. We write that*

i) $f(n) = \mathcal{O}(g(n))$ *if there exists a constant $C > 0$ and a number $n_0 \in \mathbb{N}$ such that*

$$\frac{|f(n)|}{|g(n)|} \leq C \quad \textit{for all } n > n_0$$

i.e. the growth of the function f is at most as large as that of the function g, as $n \to \infty$.

ii) $f(n) = o(g(n))$ *if for all constants $C > 0$ there exists a number n_0 such that for all $n > n_0$*

$$\frac{|f(n)|}{|g(n)|} < C \quad \textit{for all } n > n_0$$

i.e. the growth of the function f is strictly smaller than that of the function g, as $n \to \infty$ i.e. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

The Markov inequality is used in Chapter 2 to show almost sure convergence in connection with the Borel-Cantelli lemmas, below.

Lemma 5.1.2 (The Markov inequality)**.** *Let X be a real-valued random variable, $a \in (0, \infty)$ be a constant and $h : \mathbb{R} \to [0, \infty)$ a monotonically increasing function. Then it holds that*

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[h(X)]}{h(a)}. \tag{5.1.1}$$

Proof.

$$\mathbb{P}(X \geq a) = \int \mathbb{1}_{\{X \geq a\}} \mathrm{d}\mathbb{P} = \int \frac{h(X)}{h(a)} \mathrm{d}\mathbb{P} \leq \frac{\mathbb{E}[h(X)]}{h(a)}$$

□

For several results on almost sure convergence we use the famous Borel-Cantelli lemmas. To this end one needs the definition of an event occurring infinitely often.

Definition 5.1.3. *For any sequence of events $A_1, A_2, \ldots \mathcal{F}$ we say that an event A_n happens infinitely often and write A_n i.o. This occurence is also an event and we have*

$$\{A_n\ i.o.\} = \left\{ \sum_{n=1}^{\infty} \mathbb{1}_{A_n} = \infty \right\} = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k.$$

Theorem 5.1.4 (The Borel-Cantelli lemmas, see Kallenberg [50, Theorem 2.18 on p.32])**.** *Let $A_1, A_2, \ldots \mathcal{F}$ be any sequence of events. It holds that*

i)

$$\textit{If} \quad \sum_n \mathbb{P}(A_n) < \infty \quad \textit{then} \quad \mathbb{P}(\{A_n\ i.o\}).$$

ii) If the events $A_1, A_2, \ldots$ are independent then

$$\sum_n \mathbb{P}(A_n) < \infty \quad \textit{if and only if} \quad \mathbb{P}(\{A_n\ i.o\}).$$

We remind the reader of the following two classical martingale inequalities.

Theorem 5.1.5 (Burkholder's inequality, see Hall and Heyde [39, Theorem 2.10 on p.23])**.** *If $\{S_k, \mathcal{F}_k\}_{1 \leq k \leq m}$ is a martingale and $1 < p < \infty$, then there exist constants C_1 and C_2 depending only on p such that*

$$C_1 \mathbb{E}\left[\left| \sum_{k=1}^{m} X_k^2 \right|^{p/2} \right] \leq \mathbb{E}\left[|S_m|^p\right] \leq C_2 \mathbb{E}\left[\left| \sum_{k=1}^{m} X_k^2 \right|^{p/2} \right], \tag{5.1.2}$$

where $C_1 = \frac{1}{(18p^{1/2}q)^p}$, $C_2 = \left(18pq^{1/2}\right)^p$, $\frac{1}{p} + \frac{1}{q} = 1$, $X_1 = S_1$ and $X_k = S_k - S_{k-1}$, $2 \leq k \leq m$.

Theorem 5.1.6 (Doob's inequality, see Hall and Heyde [39, Theorem 2.2 on p.15])**.** *If $\{S_k, \mathcal{F}_k\}_{1 \leq k \leq m}$ is a martingale then for $p > 1$,*

$$\mathbb{E}\Big[|S_m|^p \Big]^{1/p} \leq \mathbb{E}\left[\left(\max_{1 \leq k \leq m} |S_k| \right)^p \right]^{1/p} \leq q \cdot \mathbb{E}\Big[|S_k|^p \Big]^{1/p}, \tag{5.1.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The Hájek-Rényi maximal inequality is useful to prove convergence rates in strong laws of large numbers.

Theorem 5.1.7 (A Hájek-Rényi type maximal inequality, see Fazekas and Klesov [30, Theorem 1.1 on p.437]). *Let* $X_1, X_2, \ldots$ *be a sequence of real-valued random variables and* $S_n := X_1 + \cdots + X_n$ *be their* n*:th partial sum. Suppose that* $\beta_1, \ldots, \beta_n$ *is a nondecreasing sequence of positive numbers and* $\alpha_1, \ldots, \alpha_n$ *a sequence of nonnegative numbers. Let* r *be a fixed positive number and assume that for each* m *with* $1 \leq m \leq n$

$$\mathbb{E}\left[\left(\max_{1\leq k\leq m} |S_k|\right)^r\right] \leq \sum_{k=1}^{m} \alpha_k. \tag{5.1.4}$$

Then,

$$\mathbb{E}\left[\left(\max_{1\leq k\leq m} \left|\frac{S_k}{\beta_k}\right|\right)^r\right] \leq 4\sum_{k=1}^{n} \frac{\alpha_k}{\beta_k^2}. \tag{5.1.5}$$

A practical result for the summability of weighted sums is the following theorem.

Lemma 5.1.8 (Dini Theorem, see Fichtenholz [31, Theorem 375.5 on p.304] or Shuhe and Ming [72, Lemma 1.1 on p.844]). *Let* $c_1, c_2, \ldots$ *be a sequence of nonnegative numbers and* $\nu_m := \sum_{k=m}^{\infty} c_k$*, if* $0 < \nu_m < \infty$ *for* $m \geq 1$*, then*

$$\sum_{k=1}^{\infty} \frac{c_k}{\nu_k^{\delta}} < \infty \quad \textit{for all } \delta \in (0,1). \tag{5.1.6}$$

Transforms in general, and the characteristic function in particular, are often a handy tool to show convergence in distribution.

Definition 5.1.9 (Characteristic function of a random variable). *The characteristic function* $\varphi_X : \mathbb{R}^d \to \mathbb{C}$ *of the real-valued random variable* X *is given by*

$$\varphi_X(t) = \mathbb{E}\left[e^{itX}\right]. \tag{5.1.7}$$

When considering sums of i.i.d. random variables, the following two results involving characteristic functions both hold.

Theorem 5.1.10 (See e.g. Gut [37, Theorem 4.6 and Corollary 4.6.1 on p.75]). *Let $\{X_k\}_{k\geq 1}$ be a sequence of independent random variables and $S_n := X_1 + \ldots + X_n$ be their n:th partial sum. It then holds that*

$$\varphi_{S_n}(t) = \prod_{k=1}^{n} \varphi_{X_k}(t). \tag{5.1.8}$$

Additionally, if the elements of $\{X_k\}_{k\geq 1}$ are identically distributed:

$$\varphi_{S_n}(t) = \left(\varphi_X(t)\right)^n . \tag{5.1.9}$$

The next theorem can be used to show weak convergence using the convergence of characteristic functions.

Theorem 5.1.11 (Lévy's continuity theorem, see e.g. Kallenberg [50, Theorem 4.3 on p.63]). *The sequence of random variables $\{X_n\}_{n\geq 1}$ converges weakly to the random variable X_∞:*

$$\{X_n\}_{n\geq 1} \Rightarrow X_\infty \quad \text{as } n \to \infty$$

if and only if the characteristic functions of X_n converge pointwise to the characteristic function of X_∞:

$$\varphi_{X_n}(t) = \varphi_{X_\infty}(t) \quad \text{as } n \to \infty \text{ for every } t \in \mathbb{R}^d. \tag{5.1.10}$$

The following proposition by Aldous can be used to show weak convergence for a sequence of martingale processes.

Proposition 5.1.12 (Aldous [4, Proposition 1.2 on p.587]). *Let $\{M_n(t) : t \in [0, \infty)\}_{n\geq 1}$ be a sequence of martingales. Suppose that*

i) The finite dimensional distributions of M_n converge to a limiting distribution M_∞:

$$M_n \to_{fdd} M_\infty.$$

ii) $M_\infty(t)$ is continuous in t.

iii) For each $t \in [0, \infty)$, $\{M_n(t)\}_{n\geq 1}$ is uniformly integrable.

Then,

$$M_n \Rightarrow M_\infty, \quad \textit{as } n \to \infty.$$

5.2 Appendix for Chapter 2

To calculate various convergence rates we need a Brunk-Prokhorov type law of large numbers (Theorem 5.2.2), which is a martingale difference array generalization (for the case $q = 1$ and $r = 2$) of the following powerful result by Shuhe et al.

Theorem 5.2.1 (Brunk-Prokhorov Law of Large Numbers for martingale difference sequences, see Shuhe et al. [71, Theorem 2.2 on p.3190])**.** *Let $\{X_k\}_{k\geq 1}$ be a real-valued martingale difference sequence with respect to the filtration $\{\mathcal{F}_k\}_{k\geq 1}$, where the σ-algebra is generated by the random variables $X_1, \ldots, X_k$ and $S_m := X_1 + \ldots + X_m$. Assume that $q \in (\frac{1}{2}, 1]$ and let $\{b_k\}_{k\geq 1}$ be a nondecreasing unbounded sequence. If*

$$\sum_{k=1}^{\infty} \frac{\mathbb{E}\left[|X_k|^{2q}\right]}{b_k^{2q}} < \infty,$$

then

$$\lim_{m\to\infty} \frac{S_m}{b_m} = 0 \quad \textit{a.s.}$$

with growth rate

$$\frac{S_m}{b_m} = \mathcal{O}\left(\frac{\beta_m}{b_m}\right) \quad \textit{a.s.}$$

where

$$\beta_m := \max_{1\leq k\leq m} b_k \nu_k^{\frac{\delta}{r}}, \quad \forall \delta \in (0,1), \quad \nu_m := \sum_{k=m}^{\infty} \frac{\alpha_k}{b_k^r} \quad \textit{and} \quad \lim_{k\to\infty} \frac{\beta_k}{b_k} = 0$$

for $r = 2q$, $\alpha_k = \frac{(18)^{2q}(2q)^{5q}}{(2q-1)^{3q}} \mathbb{E}\left[|X_k|^{2q}\right]$ and

$$\mathbb{E}\left[\left(\sup_{m\geq 1}\left|\frac{S_m}{b_m}\right|\right)^2\right] \leq 4\frac{(18)^{2q}(2q)^{5q}}{(2q-1)^{3q}} \sum_{k=1}^{\infty} \frac{\mathbb{E}\left[|X_k|^{2q}\right]}{b_k^{2q}} < \infty.$$

Furthermore, if one assumes that $\alpha_k > 0$ for infinitely many k, then

$$\mathbb{E}\left[\left(\sup_{m\geq 1}\left|\frac{S_m}{\beta_m}\right|\right)^2\right] \leq 4\frac{(18)^{2q}(2q)^{5q}}{(2q-1)^{3q}} \sum_{k=1}^{\infty} \frac{\mathbb{E}\left[|X_k|^{2q}\right]}{\beta_k^{2q}} < \infty.$$

Our martingale difference array generalization of the above, when $q = 1$ and $r = 2$, is the theorem below.

Theorem 5.2.2 (A Brunk-Prokhorov Strong Law of Large Numbers for martingale difference arrays)**.** *Suppose that* $\{X_k^{(n)}\}_{1\le k\le n^s, n\ge 1}$ *is a real-valued martingale difference array w.r.t. the* σ*-algebra* $\mathcal{F}_k$ *for all* $n \ge 1$ *with* $s \in [1,\infty)$ *being fixed. Let* $S_{n^s}^{(n)} := X_1^{(n)} + \ldots + X_{n^s}^{(n)}$ *and* $\{b_k\}_{k\ge 1}$ *be a non-decreasing unbounded sequence.*

If there exists a sequence $\{\alpha_k\}_{k\ge 1}$*, where* $\alpha_k > 0$ *for infinitely many* k*, such that*

$$\mathbb{E}\left[\left|X_k^{(n)}\right|^2\right] \le \alpha_k \quad \text{for all } k,\, n \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{\alpha_k}{b_k^2} < \infty, \tag{5.2.1}$$

then

$$\frac{S_{n^s}^{(n)}}{b_{n^s}} = \mathcal{O}\left(\frac{\beta_{n^s}}{b_{n^s}}\right) = o(1) \quad a.s. \tag{5.2.2}$$

where

$$\beta_m := \max_{1\le k\le m} b_k \nu_k^{\frac{\delta}{2}}, \quad \delta \in (0,1) \text{ and } \quad \nu_m := \sum_{k=m}^{\infty} \frac{\alpha_k}{b_k^2}. \tag{5.2.3}$$

More precisely, there exist a random variable $C_0 < \infty$ *a.s. and a number* $n_0 \in \mathbb{N}$ *such that*

$$\left|\frac{S_{n^s}^{(n)}}{b_{n^s}}\right| \le C_0 \frac{\beta_{n^s}}{b_{n^s}} \quad a.s. \text{ for all } n > n_0 \text{ and } \quad \frac{\beta_{n^s}}{b_{n^s}} \to 0, \quad \text{as } n \to \infty. \tag{5.2.4}$$

Proof. We first show that $\sup_{n\ge 1}\left|\frac{S_{n^s}^{(n)}}{b_{n^s}}\right| < \infty$ a.s. Since $\{X_k^{(n)}\}_{1\le k\le n^s, n\ge 1}$ is a martingale difference array, it holds that $S_{n^s}^{(n)}$ is $\mathcal{F}_{n^s}$-measurable for all n. We have that there exists a deterministic constant $C < \infty$ by Burkholder's inequality (Theorem 5.1.5) such that

$$\mathbb{E}\left[\left|S_{n^s}^{(n)}\right|^2\right] \le C \cdot \mathbb{E}\left[\left|\sum_{k=1}^{n^s}\left|X_k^{(n)}\right|^2\right|\right] = C \cdot \sum_{k=1}^{n^s} \mathbb{E}\left[\left|X_k^{(n)}\right|^2\right] \le C \cdot \sum_{k=1}^{n^s} \alpha_k \tag{5.2.5}$$

by the first part of condition (5.2.1). Using Doob's inequality (Theorem 5.1.6), one has

$$\mathbb{E}\left[\left(\max_{1\le i\le n^s}\left|S_i^{(n)}\right|\right)^2\right] \le 4\mathbb{E}\left[\left|S_{n^s}^{(n)}\right|^2\right] \quad \text{for all } n.$$

Combining this with (5.2.5) yields

$$\mathbb{E}\left[\left(\max_{1\le i\le n^s}\left|S_i^{(n)}\right|\right)^2\right] \le 4C\cdot\mathbb{E}\left[\left|\sum_{k=1}^{n^s}\left|X_k^{(n)}\right|^2\right|\right] = 4C\cdot\sum_{k=1}^{n^s}\mathbb{E}\left[\left|X_k^{(n)}\right|^2\right] \le 4C\cdot\sum_{k=1}^{n^s}\alpha_k. \tag{5.2.6}$$

Applying the Hájek-Rényi type maximal inequality of Fazekas and Klesov (Theorem 5.1.7) on (5.2.6) we get

$$\mathbb{E}\left[\left(\max_{1\le k\le n^s}\left|\frac{S_k^{(n)}}{b_k}\right|\right)^2\right] \le 4\cdot 4C\cdot\sum_{k=1}^{n^s}\frac{\alpha_k}{b_k^2} \le 16C\cdot\sum_{k=1}^{\infty}\frac{\alpha_k}{b_k^2} < \infty \tag{5.2.7}$$

by the second part of condition (5.2.1). By monotone convergence, for the supremum it implies

$$\mathbb{E}\left[\left(\sup_{n\ge 1}\left|\frac{S_{n^s}^{(n)}}{b_{n^s}}\right|\right)^2\right] = \lim_{n\to\infty}\left\{\mathbb{E}\left[\left(\max_{1\le k\le n^s}\left|\frac{S_k^{(n)}}{b_k}\right|\right)^2\right]\right\} \le 16C\cdot\sum_{k=1}^{\infty}\frac{\alpha_k}{b_k^2} < \infty \tag{5.2.8}$$

and thus $\sup_{n\ge 1}\left|\frac{S_{n^s}^{(n)}}{b_{n^s}}\right| < \infty$ a.s. Next, we prove (5.2.4) and have

$$0 \le \left|\frac{S_{n^s}^{(n)}}{b_{n^s}}\right| \le \sup_{n\ge 1}\left|\frac{S_{n^s}^{(n)}}{\beta_{n^s}}\right|\cdot\frac{\beta_{n^s}}{b_{n^s}} \tag{5.2.9}$$

so we are done if we can prove the existence of the random variable

$$C_0 := \sup_{n\ge 1}\left|\frac{S_{n^s}^{(n)}}{\beta_{n^s}}\right| < \infty \quad \text{a.s.}$$

of (5.2.4) and that $\frac{\beta_{n^s}}{b_{n^s}} \to 0$ as $n\to\infty$. By assumption $\alpha_k > 0$ for infinitely many k, the second part of (5.2.1) and the Dini theorem (Lemma 5.1.8) one has using assumption (5.2.1):

$$\sum_{k=1}^{\infty}\frac{\alpha_k}{b_k^2\nu_k^\delta} < \infty \quad \text{and thus by definition (5.2.3)} \quad \sum_{k=1}^{\infty}\frac{\alpha_k}{\beta_k^2} \le \sum_{k=1}^{\infty}\frac{\alpha_k}{b_k^2\nu_k^\delta} < \infty. \tag{5.2.10}$$

In complete analogy to (5.2.7) and (5.2.8), we have by (5.2.10) that

$$\mathbb{E}\left[\left(\max_{1\le k\le n^s}\left|\frac{S_k^{(n)}}{\beta_k}\right|\right)^2\right] \le 4\cdot 4C\cdot\sum_{k=1}^{n^s}\frac{\alpha_k}{\beta_k^2} \le 16C\cdot\sum_{k=1}^{\infty}\frac{\alpha_k}{\beta_k^2} < \infty \tag{5.2.11}$$

and

$$\mathbb{E}\left[\left(\sup_{n\ge 1}\left|\frac{S_{n^s}^{(n)}}{\beta_{n^s}}\right|\right)^2\right] = \lim_{n\to\infty}\left\{\mathbb{E}\left[\left(\max_{1\le k\le n^s}\left|\frac{S_k^{(n)}}{\beta_k}\right|\right)^2\right]\right\} \le 16C\cdot\sum_{k=1}^{\infty}\frac{\alpha_k}{\beta_k^2} < \infty. \tag{5.2.12}$$

Thus, the random variable $C_0 = \sup\limits_{n\ge 1}\left|\frac{S_{n^s}^{(n)}}{\beta_{n^s}}\right|$ exists and is almost surely finite.

Finally, $0 < \beta_m \le \beta_{m+1}$ by (5.2.3) as the maximum is taken over an increasing set.

For any $m_1 > 0$:

$$\frac{\beta_m}{b_m} \leq \frac{\max\limits_{1\leq k<m_1} b_k \nu_k^{\delta/2}}{b_m} + \frac{\max\limits_{m_1\leq k\leq m} b_k \nu_k^{\delta/2}}{b_m} \leq \frac{\max\limits_{1\leq k<m_1} b_k \nu_k^{\delta/2}}{b_m} + \nu_{m_1}^{\delta/2} \to 0 \quad \text{as } m \to \infty \tag{5.2.13}$$

since $b_m \to \infty$ and $\nu_m \to 0$, respectively. □

Remark 5.2.3. *To estimate the tail sum $\nu_m = \sum_{k=m}^{\infty} \frac{\alpha_k}{b_k^r}$ in (5.2.3), the following is useful. For f monotone decreasing and non-negative on $[m, \infty)$, one has the inequalities*

$$\int_m^\infty f(k)\mathrm{d}k \leq \sum_{k=m}^{\infty} f(k) \leq f(m) + \int_m^\infty f(k)\mathrm{d}k.$$

The following properties are easily shown for the translation operators of Definition 2.1.2.

Lemma 5.2.4 (Properties of the translation operators $T_+^{(n)}$ and $T_-^{(n)}$)**.** *The translation operators $T_+^{(n)}, T_-^{(n)} : L^2(\mathbb{R}, \mathbb{R}) \to L^2(\mathbb{R}, \mathbb{R})$ are*

i) Linear:

$$T_\pm^{(n)}(\alpha u + \beta v) = \alpha T_\pm^{(n)}(u) + \beta T_\pm^{(n)}(v), \quad \textit{for all } \alpha, \beta \in \mathbb{R} \textit{ and } u, v \in L^2(\mathbb{R}, \mathbb{R}).$$

ii) Commutative and each others inverse:

$$\left(T_+^{(n)} \circ T_-^{(n)}\right)(u) = \left(T_-^{(n)} \circ T_+^{(n)}\right)(u) = (Id)(u) = u \quad \textit{for all } u \in L^2(\mathbb{R}, \mathbb{R}).$$

iii) Isometric:

$$\|T_\pm^{(n)}(u)\|_{L^2} = \|u\|_{L^2} \quad \textit{for all } u \in L^2(\mathbb{R}, \mathbb{R}).$$

Proof. i) follows by the definition of the translation operator and the fact that $L^2(\mathbb{R}, \mathbb{R})$ is a linear space:

$$\begin{aligned} T_+^{(n)}\left((\alpha u + \beta v)(\cdot)\right) &= (\alpha u + \beta v)\left(\cdot + \Delta x^{(n)}\right) \\ &= \alpha u(\cdot + \Delta x^{(n)}) + \beta v(\cdot + \Delta x^{(n)}) \\ &= \alpha T_+^{(n)}(u) + \beta T_+^{(n)}(v) \end{aligned}$$

and analogously for the translation operator $T_-^{(n)}$.

ii) is clear since for all $u \in L^2(\mathbb{R},\mathbb{R})$

$$\begin{aligned}\left(T_+^{(n)} \circ T_-^{(n)}\right)\left(u(\cdot)\right) &= T_+^{(n)}\left(u(\cdot - \Delta x^{(n)})\right)\\ &= u(\cdot - \Delta x^{(n)} + \Delta x^{(n)}) = (Id)\,(u) = u\\ &= u(\cdot + \Delta x^{(n)} - \Delta x^{(n)})\\ &= T_-^{(n)}\left(u(\cdot + \Delta x^{(n)})\right) = \left(T_-^{(n)} \circ T_+^{(n)}\right)\left(u(\cdot)\right).\end{aligned}$$

iii) follows by the definition of the $L^2(\mathbb{R},\mathbb{R})$-norm

$$\begin{aligned}\|T_+^{(n)}(u)\|_{L^2} &= \left(\int_{-\infty}^{\infty}\left|T_+^{(n)}(u)(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}\\ &= \left(\int_{-\infty}^{\infty}\left|u(x+\Delta x^{(n)})\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty}|u(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} = \|u\|_{L^2}\\ &= \left(\int_{-\infty}^{\infty}\left|u(x-\Delta x^{(n)})\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty}\left|T_-^{(n)}(u)(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}\\ &= \|T_-^{(n)}(u)\|_{L^2}.\end{aligned}$$

□

By our definition of the mesh the initial relative volume densities $v_{i,0}^{(n)}$ (2.1.4), the discretized placement/cancelation price densities $f^{(n),I}$ (2.1.4) and the random placement/cancelation functions $g_k^{(n),I}$ for $I = C, D, G, H$ are step functions over the disjoint intervals $[x_j^{(n)}, x_{j+1}^{(n)})$, respectively. When applying the $L^2(\mathbb{R},\mathbb{R})$-norm to such a function one has

$$\begin{aligned}\|v_{b,0}^{(n)}\|_{L^2} &= \left(\int_{-\infty}^{\infty}\left|\sum_{j=-\infty}^{\infty} v_{b,0}^{(n),j}\mathbb{1}_{[x_j^{(n)},x_{j+1}^{(n)})}(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}\\ &= \left(\int_{-\infty}^{\infty}\left|\sum_{j=-\infty}^{\infty}\left\{v_{b,0}^{(n),j}\right\}\mathbb{1}_{[x_j^{(n)},x_{j+1}^{(n)})}(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}\\ &= \left(\int_{-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\left|v_{b,0}^{(n),j}\right|^2\mathbb{1}_{[x_j^{(n)},x_{j+1}^{(n)})}(x)\mathrm{d}x\right)^{\frac{1}{2}}\\ &= \left(\Delta x^{(n)}\sum_{j=-\infty}^{\infty}\left|v_{b,0}^{(n),j}\right|^2\right)^{\frac{1}{2}} \qquad (5.2.14)\end{aligned}$$

since only the quadratic terms of the squared sum remain and the cross terms are identically zero as

$$\mathbb{1}_{[x_i^{(n)},x_{i+1}^{(n)})}(x)\cdot\mathbb{1}_{[x_j^{(n)},x_{j+1}^{(n)})}(x)=\delta_{ij}\cdot\mathbb{1}_{[x_j^{(n)},x_{j+1}^{(n)})}(x), \tag{5.2.15}$$

where δ_{ij} denotes the Kronecker delta function.

The estimates below are used in Lemma 2.2.9.

Lemma 5.2.5 (Bounds and convergence of the initial volume, the placement and cancelation densities)**.** *It holds that*

i) There exist constants $\underline{K}_{v_i,0},\overline{K}_{v_i,0},\underline{K}_{f^I},\overline{K}_{f^I}\in(0,+\infty)$ *such that*

$$\underline{K}_{v_i,0}\le\|v_{i,0}^{(n)}\|_{L^2}\le\overline{K}_{v_i,0}\quad\textit{for } i\in\{b,s\}\quad\textit{and}\quad\underline{K}_{f^I}\le\|f^{(n),I}\|_{L^2}\le\overline{K}_{f^I}$$

for $I\in\{C,D,G,H\}$.

ii) The initial volume density $v_{i,0}^{(n)}$ *converges to the initital volume density* $v_{i,0}$ *of the limiting model (2.1.31)*

$$\|v_{i,0}^{(n)}-v_{i,0}\|_{L^2}=\mathcal{O}(\Delta x^{(n)})\quad\textit{for } i\in\{b,s\},$$

and the price and cancelation placement densities converge to those of the limiting model

$$\|f^{(n),I}-f^I\|_{L^2}=\mathcal{O}(\Delta x^{(n)})\quad\textit{for } I\in\{C,D,G,H\}$$

iii) The translations of the scaled initial relative buy and sell volume densities, relative price and cancelation placement densities converge for $i\in\{b,s\}$ *and the events* $I\in\{C,D,G,H\}$*:*

$$\|T_\pm^{(n)}\left(v_{i,0}^{(n)}\right)-v_{i,0}^{(n)}\|_{L^2}=\mathcal{O}(\Delta x^{(n)})\quad\textit{and}\quad\|T_\pm^{(n)}\left(f^{(n),I}\right)-f^{(n),I}\|_{L^2}=\mathcal{O}(\Delta x^{(n)}).$$

Proof. Claims i)-iii) easily follow from the definitions (2.1.4)-(2.1.5) and Assumption 2.1.8 for the initial densities and price densities by taking the norm, considering their Taylor expansions and Lagrangean remainder terms. To show the upper bound, we recall the definition of the value of $v_{i,0}^{(n),j}$ for $i\in\{b,s\}$ in (2.1.4) and the differentiability condition (2.1.5), we have

$$\begin{aligned}v_{i,0}^{(n),j}=\frac{1}{\Delta x^{(n)}}\int_{x_j^{(n)}}^{x_{j+1}^{(n)}}v_{i,0}(x)\mathrm{d}x&=\frac{V_{i,0}(x_{j+1}^{(n)})-V_{i,0}(x_j^{(n)})}{\Delta x^{(n)}}\\&=v_{i,0}(x_j^{(n)})+\frac{1}{2}v_{i,0}'(x_j^{(n)})\Delta x^{(n)}\end{aligned} \tag{5.2.16}$$

where the last term of (5.2.16) is the Lagrangean remainder term when considering the Taylor expansion of $v_{i,0}$.

It follows by (5.2.14) and substituting with (5.2.16) that

$$\begin{aligned}
\|v_{i,0}^{(n)}\|_{L^2} &= \left(\Delta x^{(n)} \sum_{j=-\infty}^{\infty} \left|v_{i,0}^{(n),j}\right|^2\right)^{\frac{1}{2}} \\
&= \left(\Delta x^{(n)} \sum_{j=-\infty}^{\infty} \left|v_{i,0}(x_j^{(n)}) + \frac{1}{2} v_{i,0}'(x_j^{(n)}) \Delta x^{(n)}\right|^2\right)^{\frac{1}{2}} \\
&= \left(\Delta x^{(n)} \sum_{j=-\infty}^{\infty} \left\{ \left|v_{i,0}(x_j^{(n)})\right|^2 + \left|v_{i,0}(x_j^{(n)})\right| \left|v_{i,0}'(x_j^{(n)})\right| \Delta x^{(n)} \right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{4}\left|v_{i,0}'(x_j^{(n)})\right|^2 (\Delta x^{(n)})^2 \right\}\right)^{\frac{1}{2}} \\
&\leq \overline{K}_{v_{i,0}}
\end{aligned} \tag{5.2.17}$$

where the upper bound $\overline{K}_{v_{i,0}}$ exists since, our partition makes the expression into a Riemann sum which converges to an improper integral as $n \to \infty$, which exists by the integrability assumptions (2.1.5).

The lower bounds $\underline{K}_{v_{i,0}}$ exist for $i \in \{b, s\}$, since we assume that the initial buy and relative volume vectors in the order book are strictly positive and square summable (2.1.5). The bounds for $f^{(n),I}$ exist by the C^2-property (2.1.28), the function attains a maximum and a minimum over the interval $[-M, M]$ and the function is identically zero outside of the interval.

We prove ii) by the definition of the scaled initial relative volume density and the $L^2(\mathbb{R})$-norm:

$$\begin{aligned}
\|v_{i,0}^{(n)} - v_{i,0}\|_{L^2} &= \left(\int_{-\infty}^{\infty} \left|\sum_{j=-\infty}^{\infty} v_{i,0}^{(n),j} \mathbb{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(x) - v_{i,0}(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&= \left(\int_{-\infty}^{\infty} \left|\sum_{j=-\infty}^{\infty} \left\{v_{i,0}^{(n),j} - v_{i,0}(x)\right\} \mathbb{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(x)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&= \left(\int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \left|v_{i,0}^{(n),j} - v_{i,0}(x)\right|^2 \mathbb{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(x) \mathrm{d}x\right)^{\frac{1}{2}}
\end{aligned} \tag{5.2.18}$$

since only the quadratic terms of the squared sum remain and the cross terms are identically zero since the intervals are disjoint.

For the terms $\left|v_{i,0}^{(n),j} - v_{i,0}(x)\right|^2$ of (5.2.18) on the interval $[x_j^{(n)}, x_{j+1}^{(n)})$ we have for the unsquared term by (5.2.16) and using the mean value theorem for differentiable functions that

$$\begin{aligned}\left|v_{i,0}^{(n),j} - v_{i,0}(x)\right| &= \left|v_{i,0}(x_j^{(n)}) + \frac{1}{2}v'_{i,0}(x_j^{(n)})\Delta x^{(n)} - v_{i,0}(x)\right| \\ &\leq \left|v_{i,0}(x_j^{(n)}) - v_{i,0}(x)\right| + \frac{1}{2}\left|v'_{i,0}(x_j^{(n)})\right|\Delta x^{(n)} \\ &= \Bigg\{\text{since } \left|v_{i,0}(x_j^{(n)}) - v_{i,0}(x)\right| = \left|v'_{i,0}(c_j^{(n)})\right|\Delta x^{(n)} \text{ for some} \\ &\qquad c_j^{(n)} \in (x_j^{(n)}, x_{j+1}^{(n)}) \text{ by the mean-value theorem}\Bigg\} \\ &= \left|v'_{i,0}(c_j^{(n)})\right|\Delta x^{(n)} + \frac{1}{2}\left|v'_{i,0}(x_j^{(n)})\right|\Delta x^{(n)} \\ &= \Delta x^{(n)}\left(\left|v'_{i,0}(c_j^{(n)})\right| + \frac{1}{2}\left|v'_{i,0}(x_j^{(n)})\right|\right) \end{aligned} \tag{5.2.19}$$

Substituting the upper bound (5.2.19) into (5.2.18), we get an estimate over step functions

$$\begin{aligned}&\|v_{i,0}^{(n)} - v_{i,0}\|_{L^2} \\ &\leq \left(\int_{-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\left\{\left(\Delta x^{(n)}\right)^2\left||v'_{i,0}(c_j^{(n)})| + \frac{1}{2}|v'_{i,0}(x_j^{(n)})|\right|^2\right\}\mathbb{1}_{[x_j^{(n)}, x_{j+1}^{(n)})}(x)\mathrm{d}x\right)^{\frac{1}{2}} \\ &= \Delta x^{(n)}\cdot\underbrace{\left(\Delta x^{(n)}\sum_{j=-\infty}^{\infty}\left\{|v'_{i,0}(c_j^{(n)})|^2 + |v'_{i,0}(x_j^{(n)})||v'_{i,0}(c_j^{(n)})| + \frac{1}{4}|v'_{i,0}(x_j^{(n)})|^2\right\}\right)^{\frac{1}{2}}}_{\leq \overline{K}_{\Delta v_{i,0}}} \\ &= \mathcal{O}(\Delta x^{(n)}).\end{aligned}$$

where the upper bound $\overline{K}_{\Delta v_b}$ exists by the analogous existence arguments of (5.2.17).

For the relative placement and cancelation densities $f^{(n),I}$ for $I \in \{C, D, G, H\}$, the claim follows by the analogous argument used in the proof of i).

We now prove iii), again arguing for the initial relative volume density $v_{i,0}^{(n)}$ and we have by (5.2.14) that

$$\|T_+^{(n)}\left(v_{i,0}^{(n)}\right) - v_{i,0}^{(n)}\|_{L^2} = \left(\Delta x^{(n)} \sum_{j=-\infty}^{\infty} \left|v_{i,0}^{(n),j+1} - v_{i,0}^{(n),j}\right|^2\right)^{\frac{1}{2}} \tag{5.2.20}$$

For the term $\left|v_{i,0}^{(n),j+1} - v_{i,0}^{(n),j}\right|^2$, we get, by the definition of the initial relative volume density and the differential quotient of the second derivative, a Lagrangean remainder term since $v_{i,0}$ was assumed to be C^2:

$$\begin{aligned}
\left|v_{i,0}^{(n),j+1} - v_{i,0}^{(n),j}\right| &= \left|\frac{1}{\Delta x^{(n)}}\left(\int_{x_{j+1}^{(n)}}^{x_{j+2}^{(n)}} v_{i,0}(x)\mathrm{d}x - \int_{x_j^{(n)}}^{x_{j+1}^{(n)}} v_{i,0}(x)\mathrm{d}x\right)\right| \\
&= \left|\frac{V_{i,0}(x_{j+2}^{(n)}) - 2V_{i,0}(x_{j+1}^{(n)}) + V_{i,0}(x_j^{(n)})}{\Delta x^{(n)}}\right| \\
&= \left|v_{i,0}'(x_{j+1}^{(n)})\Delta x^{(n)} + \frac{1}{2}v_{i,0}''(x_{j+1}^{(n)}) \cdot (\Delta x^{(n)})^2\right| \\
&= \Delta x^{(n)}\left|v_{i,0}'(x_{j+1}^{(n)}) + \frac{1}{2}v_{i,0}''(x_{j+1}^{(n)})\Delta x^{(n)}\right|.
\end{aligned} \tag{5.2.21}$$

Substituting (5.2.21) into (5.2.20) we get

$$\begin{aligned}
&\|T_+^{(n)}\left(v_{i,0}^{(n)}\right) - v_{i,0}^{(n)}(\cdot)\|_{L^2} \\
&= \left(\Delta x^{(n)} \sum_{j=-\infty}^{\infty} \left(\Delta x^{(n)}\right)^2 \left|v_{i,0}'(x_{j+1}^{(n)}) + \frac{1}{2}v_{i,0}''(x_{j+1}^{(n)})\Delta x^{(n)}\right|^2\right)^{\frac{1}{2}} \\
&= \Delta x^{(n)} \underbrace{\left(\Delta x^{(n)} \sum_{j=-\infty}^{\infty} \left\{\left|v_{i,0}'(x_{j+1}^{(n)})\right|^2 + v_i'(x_{j+1}^{(n)})v_{i,0}''(x_{j+1}^{(n)})\Delta x^{(n)} + \left|v_{i,0}''(x_{j+1}^{(n)})\Delta x^{(n)}\right|^2\right\}\right)^{\frac{1}{2}}}_{\leq \overline{K}_{T_+ v_{i,0}}} \\
&= \mathcal{O}(\Delta x^{(n)}).
\end{aligned}$$

where the upper bound $\overline{K}_{T_+ v_{i,0}}$ again exists by the analogous existence arguments of (5.2.17).

One can use the analogous arguments for the quantity $\|T_-^{(n)}\left(v_{i,0}^{(n)}\right) - v_{i,0}^{(n)}(\cdot)\|_{L^2}$.

Again, the corresponding bounds $\overline{K}_{T_- f^I}$ for the relative placement and cancelation densities $f^{(n),I}$ for $I \in \{C, D, G, H\}$ with corresponding upper bounds exist by differentiability and compactness arguments.

Thus, iii) is proved. □

In the second part of Chapter 2, the concept of triangular martingale difference arrays is used.

Definition 5.2.6. *A family of random variables y_k^n, $k = 1, ..., n$, $n = 1, 2, ...$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a triangular martingale difference array (TMDA) with respect to a family $\{\mathcal{F}^n\}_{n=1,2,...}$ of filtrations, $\mathcal{F}^n = \{\mathcal{F}_k^n\}_{k=1}^n$, if for all $n = 1, 2, ...$ the sequence $y_1^n, ..., y_n^n$ is an $\mathcal{F}^n$-martingale difference sequence (MDS), i.e.*

$$\mathbb{E}\left[y_k^n | \mathcal{F}_{k-1}^n\right] = 0.$$

To prove the WLLN for LOBs, the following moment estimate is very useful.

Lemma 5.2.7 (Martingale estimate on Banach space, Pisier [63, p.221])**.** *Let E be a real separable p-uniformly smooth Banach space $(1 \leq p \leq 2)$. Then, for all $r \geq 1$ there exists a constant $C > 0$ such that for all martingales*

$$\left\{\left(\sum_{i=1}^n X_i, \mathcal{G}_n\right)\right\}_{n\geq 1}$$

with values in E, we have

$$\mathbb{E}\left[\sup_{n\geq 1}\left|\sum_{i=1}^n X_i\right|\right]^r \leq C\mathbb{E}\left(\sum_{n=1}^\infty |X_i|^p\right)^{r/p}.$$

This lemma allows us to prove the following law of large numbers for TMDAs.

Theorem 5.2.8. *Let y_k^n, $k = 0, ..., n$, $n = 1, 2, ...$ be a TMDA taking values in a real separable p-uniformly smooth Banach space E for $1 \leq p \leq 2$ such that*

$$\sup_{n,k} \mathbb{E}|y_k^n|^p < \infty.$$

Then, for all $\alpha > 0$ such that $\alpha \cdot p > 1$ one has for all $\epsilon > 0$ that

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\leq m\leq n} |\sum_{k=0}^m y_k^n| \geq \epsilon \cdot n^\alpha\right) = 0.$$

Proof. By Markov's inequality

$$\mathbb{P}\left(\sup_{0\leq m\leq n} |\sum_{k=0}^m y_k^n| \geq \epsilon \cdot n^\alpha\right) \leq \frac{1}{\epsilon n^{q\cdot\alpha}}\mathbb{E}\left[\sup_{0\leq m\leq n} |\sum_{k=0}^m y_k^n|\right]^q$$

for all $q \geq 1$. Thus, it follows from Lemma 5.2.7 that

$$\begin{aligned}\mathbb{P}\left(\sup_{0\leq m\leq n} |\sum_{k=0}^{m} y_k^n| \geq \epsilon \cdot n^{\alpha}\right) &\leq Cn^{-\alpha\cdot q}\mathbb{E}\left[\sum_{k=0}^{n} |y_k^n|^p\right]^{q/p} \\ &\leq Cn^{-\alpha\cdot q+\frac{q}{p}}\end{aligned}$$

for a generic constant $C > 0$ since the random variables y_k^n have a uniformly bounded p:th moment. Hence, the assertion follows as soon as $-\alpha \cdot q + \frac{q}{p} < 0$, which holds for all $q > 0$ as $\alpha \cdot p > 1$. □

As an immediate corollary from the preceding theorem one obtains the following law of large numbers for TMDAs.

Corollary 5.2.9. *Let y_k^n, $k = 0, ..., n$, $n = 1, 2, ...$ be a TMDA taking values in a real separable 2-uniformly smooth Banach space E such that*

$$\sup_{n,k}\left(n^{2\alpha}\mathbb{E}|y_k^n|^2\right) < \infty$$

for some $\alpha > \frac{1}{2}$. Then

$$\lim_{n\to\infty} \sup_{0\leq m\leq n} |\sum_{k=0}^{m} y_k^n| = 0 \quad \textit{in probability.}$$

If the stronger condition

$$\sup_{k}\left(n \cdot \mathbb{E}|y_k^n|^2\right) \leq \frac{C}{n^{1+\epsilon}} \tag{5.2.22}$$

for some $0 < C < \infty$ and $\epsilon > 0$, then the limit above hold almost surely.

Proof. In order to prove the first assertion, we apply Theorem 5.2.8 to the TMDA

$$\hat{y}_k^n := n^{\alpha} y_k^n.$$

Then

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{0\leq m\leq n} |\sum_{k=0}^{m} \hat{y}_k^n| \geq \epsilon \cdot n^{\alpha}\right) = 0$$

and

$$\mathbb{P}\left(\sup_{0\leq m\leq n} |\sum_{k=0}^{m} \hat{y}_k^n| \geq \epsilon \cdot n^{\alpha}\right) = \mathbb{P}\left(\sup_{0\leq m\leq n} |\sum_{k=0}^{m} y_k^n| \geq \epsilon\right) = 0.$$

If the stronger condition (5.2.22) holds, then the *proof* of Theorem 5.2.8 (for $\alpha = \frac{1}{2}$) shows that

$$\mathbb{P}\left(\sup_{0\leq m\leq n} |\sum_{k=0}^{m} y_k^n| \geq \epsilon\right) \leq \frac{C}{n^{1+\epsilon}}$$

and the assertion follows from the Borel-Cantelli lemma. □

In the following, we prove some properties of the volume density functions for the sequence of models defined in Section 2.3. In particular, we show that the sequences $\{\eta^{(n)}_{v,k}\}$ take values in L^2 almost surely.

Using the isometry property of the translation operator we deduce that the L^2 norm of the volume density function can be estimated from above by considering a model with only passive order placements. In such a case

$$\begin{aligned}\eta^{(n)}_{v_b,k+1} &= \eta^{(n)}_{v_b,0} + \frac{\Delta v^{(n)}}{\Delta x^{(n)}} \sum_{l=0}^{k} M_l^{(n),D} \\ &= \eta^{(n)}_{v_b,0} + \frac{\Delta v^{(n)}}{\Delta x^{(n)}} \sum_{l=0}^{k} \left(M_l^{(n),D} - \mathbb{E}[M_l^{(n),D}] \right) + \frac{\Delta v^{(n)}}{\Delta x^{(n)}} \sum_{l=0}^{k} \mathbb{E}[M_l^{(n),D}]\end{aligned}$$

and so uniformly in $k = 0, ..., \frac{T}{\Delta t^{(n)}}$ (up to some additive constant that converges to zero in the L^2-norm):

$$\|\eta^{(n)}_{v_b,k+1}\|_{L^2} \leq \|\eta^{(n)}_{v_b,0}\|_{L^2} + \Delta v^{(n)} \left\| \sum_{l=0}^{k} \left(\frac{1}{\Delta x^{(n)}} M_l^{(n),D} - f_l^{(n),D} \right) \right\|_{L^2} + T\|f^D\|_{L^2}. \tag{5.2.23}$$

With this, we the following L^2-boundedness of the volume density functions follows.

Lemma 5.2.10. *The expected L^2-norm of the volume density function is uniformly bounded:*

$$\max\{\mathbb{E}\|\eta^{(n)}_{v_b,k}\|_{L^2}, \mathbb{E}\|\eta^{(n)}_{v_b,k}\|^2_{L^2}\} \leq C \tag{5.2.24}$$

for some constant $C < \infty$ that does not depend neither on $n \in \mathbb{N}$, nor on $k = 0, ..., \lfloor T/\Delta t^{(n)} \rfloor$.

Proof. The result follows from part i) of Lemma 2.2.9. □

The next result will be used to prove a Lipschitz continuity property of the grid-point approximation of the limiting PDE.

Lemma 5.2.11. *There exists a constant $C < 0$ such that for all $n \in \mathbb{N}$ and $k = 0, ..., \lfloor \frac{T}{\Delta t^{(n)}} \rfloor$*

$$\left\| \mathbb{E}\left[T_\pm^{(n)}(\eta^{(n)}_{v_b,k}) - \eta^{(n)}_{v_b,k} \right] \right\|_{L^2} \leq C \cdot \Delta x^{(n)}$$

Proof. Using the induction formula (2.2.80) of and the linearity of the translation operator $T_+^{(n)}$ it follows that

$$\begin{aligned}
&T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right)-\eta_{v_b,k}^{(n)}\\
&=\left(\left(T_+^{(n)}\right)^{\sum_{i=0}^{k-1}\mathbf{1}_i^A}\circ\left(T_-^{(n)}\right)^{\sum_{i=0}^{k-1}\mathbf{1}_i^B}\right)\left(T_+^{(n)}\left(v_{b,0}^{(n)}\right)-v_{b,0}^{(n)}\right)\\
&-\Delta v^{(n)}\cdot\sum_{i=0}^{k-1}\left(\left(T_+^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbf{1}_j^A}\circ\left(T_-^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbf{1}_j^B}\right)\left(\left[T_+^{(n)}\left(M_i^{(n),C}\eta_{v_b,i}^{(n)}\right)-M_i^{(n),C}\eta_{v_b,i}^{(n)}\right]\mathbf{1}_i^C\right)\\
&+\Delta v^{(n)}\cdot\sum_{i=0}^{k-1}\left(\left(T_+^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbf{1}_j^A}\circ\left(T_-^{(n)}\right)^{\sum_{j=i+1}^{k-1}\mathbf{1}_j^B}\right)\left(\left[T_+^{(n)}\left(M_i^{(n),D}\right)-M_i^{(n),D}\right]\mathbf{1}_i^D\right)\quad\text{a.s.}
\end{aligned}\tag{5.2.25}$$

Taking the expected value and norms in (5.2.25) we find:

$$\begin{aligned}
&\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right)-\eta_{v_b,k}^{(n)}\right]\right\|_{L^2}\\
&\leq\left\|T_+^{(n)}\left(v_{b,0}^{(n)}\right)-v_{b,0}^{(n)}\right\|_{L^2}+\sum_{i=0}^{k-1}\left\|T_+^{(n)}\left(\mathbb{E}\left[M_i^{(n),C}\eta_{v_b,i}^{(n)}\right]\right)-\mathbb{E}\left[M_i^{(n),C}\eta_{v_b,i}^{(n)}\right]\right\|_{L^2}\\
&\qquad+\Delta v^{(n)}\sum_{i=0}^{k-1}\left\|T_+^{(n)}\left(f^{(n),D}\right)-f^{(n),D}\right\|_{L^2}\\
&\leq\left\|T_+^{(n)}\left(v_{b,0}^{(n)}\right)-v_{b,0}^{(n)}\right\|_{L^2}+\Delta v^{(n)}\sum_{i=0}^{k-1}\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,i}^{(n)}\right)-\eta_{v_b,i}^{(n)}\right]\right\|_{L^2}\\
&\qquad+\Delta v^{(n)}\sum_{i=0}^{k-1}\left\|T_+^{(n)}\left(f^{(n),D}\right)-f^{(n),D}\right\|_{L^2}
\end{aligned}\tag{5.2.26}$$

Since the L^2-norms of the volume density functions are uniformly bounded, we use Assumptions 2.3.2 and 2.3.3 to deduce that for some generic constant $K>0$:

$$\begin{aligned}
\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right)-\eta_{v_b,k}^{(n)}\right]\right\|_{L^2}&\leq K\Delta x^{(n)}+\Delta v^{(n)}\sum_{i=0}^{k-1}\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,i}^{(n)}\right)-\eta_{v_b,i}^{(n)}\right]\right\|_{L^2}+K\Delta x^{(n)}\\
&\leq K\Delta x^{(n)}+\Delta v^{(n)}\sum_{i=0}^{\lfloor\frac{T}{\Delta t^{(n)}}\rfloor}\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,i}^{(n)}\right)-\eta_{v_b,i}^{(n)}\right]\right\|_{L^2}
\end{aligned}\tag{5.2.27}$$

Hence, it follows from the discrete Gronwall Lemma 4.2.8 that

$$\sup_{k=0,\ldots,\frac{T}{\Delta t^{(n)}}}\left\|\mathbb{E}\left[T_+^{(n)}\left(\eta_{v_b,k}^{(n)}\right)-\eta_{v_b,k}^{(n)}\right]\right\|_{L^2}\leq K\Delta x^{(n)}.$$

□

Corollary 5.2.12. *There exists a constant $C > 0$ such that*

$$\|u^{(n)}(\cdot + \Delta x^{(n)}, t) - u^{(n)}(\cdot, t)\|_{L^2} \leq C \cdot \Delta x^{(n)}.$$

Moreover,

$$\sup_{n \in \mathbb{N}, t \in [0,T]} \|u^{(n)}(\cdot, t)\|_{L^2} < \infty.$$

Proof. In order to establish the first assertion we represent the functions $u^{(n)}$ as

$$u_k^{(n)} = \mathbb{E}\left[\zeta_k^{(n)}\right] \tag{5.2.28}$$

where

$$\begin{cases} \zeta_{k+1}^{(n)} & := \zeta_k^{(n)} + \mathcal{D}_{v,k}^{(n)}\left(\gamma(t_k^{(n)}), \zeta_k^{(n)}\right) \\ & \\ \zeta_0^{(n),j} & := v_0^{(n),j} \end{cases}. \tag{5.2.29}$$

For $t \in [k \cdot \Delta t^{(n)}, (k+1) \cdot \Delta^{(n)})$ the preceding lemma then implies:

$$\|u^{(n)}(\cdot \pm \Delta x^{(n)}, t) - u^{(n)}(\cdot, t)\|_{L^2} = \left\|\mathbb{E}\left[T_{\pm}^{(n)}\left(\zeta_k^{(n)}\right) - \zeta_k^{(n)}\right]\right\|_{L^2} \leq C \Delta x^{(n)}.$$

The second assertion follows from (5.2.28) together with Lemma 5.2.10:

$$\|u^{(n)}(\cdot, t_k^{(n)})\|_{L^2} = \|\mathbb{E}\left[\zeta_k^{(n)}\right]\|_{L^2} \leq \mathbb{E}\left[\|\zeta_k^{(n)}\|_{L^2}\right] \leq C.$$

□

5.3 Appendix for Chapter 3

We remind the reader of the following definition.

Definition 5.3.1 (Process of locally bounded variation)**.** *A stochastic process X is said to be of locally bounded variation if its total variation process:*

$$\mathcal{V}(X)(t) := \|X(0)\| + \sup\left\{\sum_{i=1}^{l} \|X(t_i) - X(t_{i-1})\| \mid 0 = t_0 < t_1 < \cdots < t_l = t, \quad l \geq 0\right\}$$

is almost surely finite for all $t \in [0,T]$, $T \in (0, \infty)$ i.e. on each compact time interval.

By a White Noise, we mean the following Gaussian process indexed by the Borel sets.

Definition 5.3.2 (White Noise on $\mathbb{R}^d$, see Khoshnevisan [52, Theorem 1.3.1 on p.142])**.** *A Gaussian process* $\{\mathbb{W}(A) \mid A \in \mathcal{B}(\mathbb{R}^d)\}$ *with mean zero i.e.* $\mathbb{E}[\mathbb{W}(A)] = 0$ *for all* $A \in \mathcal{B}(\mathbb{R}^d)$ *and covariance*

$$\mathbb{E}[\mathbb{W}(A)\mathbb{W}(B)] = \lambda(A \cap B), \quad \text{for all } A, B \in \mathcal{B}(\mathbb{R}^d),$$

where λ *is the Lebesgue measure on* $\mathbb{R}^d$*, is called a White Noise process.*

With the above definition as a starting point, one can show a useful representation of the Brownian sheet i.e. a d-parameter Brownian motion which, equivalently to Definition 3.1.9 for $d = 2$, can be defined as follows.

Definition 5.3.3 (One dimensional Brownian sheet)**.** *The real-valued one-dimensional Gaussian process* $\{W(t) : t \in \mathbb{R}_+^d\}$ *with mean zero and covariance function*

$$\mathbb{E}[W(s)W(t)] = \prod_{i=1}^{d} (s_i \wedge t_i), \quad \text{for all } s = (s_1, \ldots, s_d), t = (t_1, \ldots, t_d) \in \mathbb{R}^d,$$

is called a d-parameter Brownian Sheet or a d-parameter Brownian motion.

The multi-parameter Brownian motions and sheets can actually be seen as a special case of the White Noise process.

Theorem 5.3.4 (Čentsov's Representation, see Khoshnevisan [52, Theorem 1.5.1 on p.148])**.** *Let* $\{\mathbb{W}(A) \mid A \in \mathcal{B}(\mathbb{R}^d)\}$ *be a white noise on* $\mathbb{R}^d$*. Then, the process* $\{W(t) : t \in \mathbb{R}_+^d\}$ *with* $W(t) = \mathbb{W}([0, t])$ *for* $t \in \mathbb{R}_+^d\}$ *is a d-dimensional Brownian sheet/Brownian motion.*

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List of Symbols and Notation

The subscript k indicates that the variable after k events, the superscript (n) that the variable is in the n:th model, the superscript $I = A, \dots, H$ that event I, the superscript j that the value at the j:th tick and the subscript b $[s]$ that the buy [sell] side is considered. This convention applies to processes throughout the thesis.

Ω	sample space
$\mathcal{F}$	σ-algebra
$\mathbb{P}$	probability measure
$\mathbb{E}$	expectation operator with respect to $\mathbb{P}$
a.s.	almost surely i.e. with probability 1
i.o.	infinitely often
$\mathbb{N}_0$	set of natural numbers $\{0, 1, 2, \dots\}$
$\mathbb{N}$	set of natural numbers $\{1, 2, \dots\}$
$\mathbb{Z}$	set of integers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_{>0}$	set of positive real numbers
$\mathbb{R}_{\geq 0}$	set of non-negative real numbers
$\mathcal{O}$ $[o]$	big [small] ordo
càdlàg	continue à droite, limite à gauche i.e. right continuous with left limits
$\to$	converges to
$\Rightarrow$	converges weakly to
LOB	limit order book
SLLN	strong law of large numbers
FCLT	functional central limit theorem
SRBM	semimartingale reflecting Brownian motion
$L^2(A, B)$	equivalence class of the set of square-integrable functions from the set A to the set B
$C^{m,n}(A, B)$	set of m times in the first variable and n times in the second variable continuously differentiable functions from the set A to the set B
$\mathcal{B}(A)$	Borel set of the the set A

$D(A, B)$	Skorokhod space i.e. the càdlàg functions from the set A to the set B
Δx	price tick
Δt	time tick
Δv	volume tick
E	state space
$\\| \cdot \\|_E$	norm of the normed space E
$\\|X\\|_r$	r:th expectation norm of X, i.e. $(\mathbb{E}[\\|X\\|^r])^{1/r}$
$\lvert \cdot \rvert$	Euclidean norm
τ_k	random clock/physical time after k events (in Chapter 2)
τ	translation operator (in Chapter 3)
C_k	random event time operator
S_k	random state after k events
$\mathcal{D}_k$	random state operator
$\mathbb{M}_{\text{buy}}$ [$\mathbb{M}_{\text{sell}}$]	buy [sell] part of the state operator $\mathcal{D}_k$
B_k [A_k]	best bid [ask] price after k events
$W^i_{b,k}$ [$W^i_{s,k}$]	buy [sell] volume at price $i\Delta x$
$V^i_{b,k}$ [$V^i_{s,k}$]	relative buy [sell] volume at i price ticks from B_k [A_k]
$v_{b,k}$ [$v_{s,k}$]	relative buy [sell] volume density
I	event index i.e. $I \in \{A, B, C, D, E, F, G, H\}$
p^I	probability that event I occurs
ϕ^I_k	inter-arrival time for event I
π^I_k	relative placement/cancelation price for event I
f^I	probability density π^I_k
ω^I_k	placed volume/cancelation proportion for event I
$\mathbb{1}^I_k$	event indicator of event I
T_+, T_-	Translation operators
$M^I_{v,k}$	relative volume density change in event I
μ_X	mean, the expectation of X
μ	time process, random in state and deterministic in time (in Chapters 2 and 4)
$\mu^\#$	drift vector of SRBM (in Chapter 3)
$\mathcal{H}$	numerical scheme operator
$s(t)$	state of limiting model at time t
$b(t)$ [$a(t)$]	best bid [ask] price in the limiting model $s(t)$
$b^*(\cdot)$	limit of the expected state operator
$m^*(\cdot)$	limit of the expected time operator
γ	vector containing best bid and ask price (in Chapter 2)

γ^i	reflection vector of boundary i (in Chapter 3)
$v_b(\cdot,t)$ $[v_s(\cdot,t)]$	relative buy [sell] volume density in the limiting model $s(t)$
$v(\cdot,t)$	vector containing relative buy and sell volume densities of the limiting model $s(t)$
Γ	vector containing the best bid and ask price (in Chapter 2)
Γ $[\Gamma^{\#}]$	covariance matrix of SRBM without [with] drift (in Chapter 3)
ν	initial distribution of SRBM
$\eta^{(n)}$	state process, random in state and deterministic in time of the n:th model
$\eta_\gamma^{(n)}$ $[\eta_v^{(n)}]$	price [volume] component of $\eta^{(n)}$
$\mathcal{I}$	index set
G_i	domains in $\mathbb{R}^d$, for $i \in \mathcal{I}$
∂G $[\partial G_i]$	boundary of G $[G_i]$
$G = \bigcap_{i\in\mathcal{I}} G_i$	domain of the SRBM
$\overline{G}$	closure of G
$\mathbf{0}$	zero vector
$\mathbf{I}$	cardinality of the index set $\mathcal{I}$, i.e. $\mathbf{I} = \lvert\mathcal{I}\rvert$
ξ_k^j	random field component for the underlying volume fluctuations at price level j after k events
X	price process further away from the boundary
Y^i	reflection process of boundary i
$W(x,t)$	2-parameter Brownian motion/Brownian sheet with price variable x and time variable t
$\mathcal{V}(X)(t)$	total variation process of the stochastic process $X(t)$
H	Hilbert space
$\langle\cdot,\cdot\rangle$	inner product of H
$L(H,H)$	set of linear operators from H to H
$\mathcal{S}$	covariance operator
W_H	Brownian motion on H
α_m	strong mixing coefficient
ϕ_m	uniform mixing coefficient
$a \wedge b$	minimum of a and b
$\mathbb{W}$	White Noise on $\mathbb{R}^d$

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Selbstständigkeitserklärung

Ich versichere hiermit, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der erlaubten Hilfsmittel angefertigt habe.

Berlin, den 05. März 2014 Michael Christoph Paulsen